Stable Enrichment and Local Preconditioning in the Particle–Partition of Unity Method

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Summary. This paper is concerned with the stability and approximation properties of enriched meshfree methods. In particular we focus on the particle-partition of unity method (PPUM) yet the presented results hold for any partition of unity based enrichment scheme.

The goal of our enrichment scheme is to recover the optimal convergence rate of the uniform h-version independent of the regularity of the solution. Hence, we employ enrichment not only for modeling purposes but rather to improve the approximation properties of the numerical scheme. To this end we enrich our PPUM function space in an enrichment zone hierarchically near the singularities of the solution. This initial enrichment however can lead to a severe ill-conditioning and can compromise the stability of the discretization. To overcome the ill-conditioning of the enriched shape functions we present an appropriate local preconditioner which yields a stable and optimally conditioned basis independent of the employed initial enrichment. The construction of this preconditioner is of linear complexity with respect to the number of discretization points.

We obtain optimal error bounds for an enriched PPUM discretization with local preconditioning that are independent of the regularity of the solution globally and within the employed enrichment zone we attain a kind of super-convergence.

The results of our numerical experiments clearly show that our enriched PPUM with local preconditioning recovers the optimal convergence rate of $O(h^p)$ of the uniform h-version globally. For the considered model problems from linear elastic fracture mechanics we obtain an improved convergence rate of $O(h^{p+\delta})$ with $\delta \geq \frac{1}{2}$ for $p = 1$. The condition number of the stiffness matrix is independent of the employed enrichment zone.

Key words: stability, meshfree method, partition of unity method, extrinsic enrichment, preconditioning

1 Introduction

Meshfree methods (MM) and other generalizations of the classical finite element method (FEM) have become accepted and fairly widely used tools in

e.g. computational mechanics in recent years. Especially the flexibility of these methods with respect to the employed shape functions lead to this development. The concept of so-called extrinsic enrichment for instance is employed in many MM [5,19,21], the extended finite element method (XFEM) [4,6,17], and the generalized finite element method (GFEM) [10–12]. Here, the fundamental assumption is the availability of a partition of unity (PU) in the discretization space $V \subset H^k(\Omega)$. In the XFEM and the GFEM the PU $\{\phi_i\}$ is derived from a mesh whereas in MM such as the particle–partition of unity method (PPUM) [13,14,20] the PU is derived from discretization points only. With the help of the PU we can easily incorporate a priori knowledge about special behavior of the sought solution $u \in H^k(\Omega)$ in the discretization space. In fracture mechanics for instance such a priori information is readily available. The solution u is discontinuous across a crack and particular singularities are located at the crack tips, i.e. $u \approx \hat{\eta}_{\text{tip}}$ with $\hat{\eta}_{\text{tip}} \in H^1(\Omega)$ known. Hence, an approximation with respect to an enriched discretization space

$$V_E := V + \sum_{i=1}^{N} \phi_i \hat{\eta}_{\text{jump}} + \sum_{i=1}^{N} \phi_i \hat{\eta}_{\text{tip}} \tag{1.1}$$

can yield a substantially better approximation than an approximation in V only. A uniform h-refinement of V_E for instance is globally consistent of an order that is *not* limited by the regularity of the solution u. However, the stability of the discretization using V_E instead of V may be substantially compromised due to the enrichment since the decomposition (1.1) is *not* a direct splitting. It is exactly this property that leads to the conditioning problems observed in enriched discretization approaches, see e.g. [7].

In this paper we present an automatic algebraic approach for the construction of a direct splitting

$$V_E = V \oplus \sum_{i=1}^{N} \phi_i \eta_{\text{jump}} \oplus \sum_{i=1}^{N} \phi_i \eta_{\text{tip}} \tag{1.2}$$

which yields a *stable* discretization *independent* of the employed initial enrichment functions $\hat{\eta}_{\text{jump}}$ and $\hat{\eta}_{\text{tip}}$; i.e., we construct a basis transformation from the instable representation (1.1) of V_E to the stable representation (1.2). Essentially we develop an appropriate preconditioner for V_E. Since the proposed scheme attains a stable discretization independent of the employed enrichment functions we obtain an optimal global convergence rate of the uniform h-version that is not limited by the regularity of the solution. Moreover, we show that we obtain an even better convergence near the singular points of the solution.

We present our approach in the context of the PPUM however it is applicable to any PU based enrichment technique. The only prerequisite of the proposed scheme is that the employed PU satisfies the flat-top property [16, 21]. This assumption is essential to maintain the stability and at the same time

allows to construct the stable splitting (1.2) by local operations in linear complexity.

The remainder of this paper is organized as follows. In Section 2 we give a short review of the essential ingredients of the multilevel PPUM. In Section 3 we introduce our enrichment scheme and the construction of a direct splitting for an enriched PPUM space which yields a stable basis independent of the employed initial enrichment functions. We show that we recover the optimal convergence behavior of the uniform h-version of the enriched PPUM independent of the regularity of the solution. Moreover, we obtain an even *better* error bound for the proposed scheme *near* the singularities of the solution. The results of our numerical experiments are given in Section 4. These results clearly show that we obtain an optimal convergence behavior of the uniform h-version of the PPUM globally and that the condition number of the stiffness matrix does not suffer from the employed enrichment. Near the singular points of the solution we obtain an almost quadratic convergence in the energy-norm using linear polynomials only. Hence, the coefficients of the asymptotic expansion of the solution, i.e. the stress intensity factors, can be computed with much higher accuracy close to the singularities of the solution. Finally, we conclude with some remarks in Section 5.

2 Particle–Partition of Unity Method

In this section let us shortly review the core ingredients of the PPUM, see [13, 14, 20] for details. In a first step, we need to construct a PPUM space V^{PU}, i.e., we need to specify the PPUM functions $u^{\mathrm{PU}} \in V^{\mathrm{PU}}$. An arbitrary function $u^{\mathrm{PU}} \in V^{\mathrm{PU}}$ is defined as the linear combination

$$u^{\mathrm{PU}}(x) = \sum_{i=1}^{N} \varphi_i(x) u_i(x) \quad \text{with} \quad u_i(x) = \sum_{m=1}^{d_i} u_i^m \vartheta_i^m(x) \tag{2.1}$$

and the respective PPUM space V^{PU} is defined as

$$V^{\mathrm{PU}} := \sum_{i=1}^{N} \varphi_i V_i \quad \text{with} \quad V_i := \operatorname{span}\langle \vartheta_i^m \rangle. \tag{2.2}$$

Here, we assume that the functions φ_i form a partition of unity (PU) on the domain Ω and refer to the spaces V_i with $\dim(V_i) = d_i$ as local approximation spaces. Hence, the shape functions employed in the PPUM are the products $\varphi_i \vartheta_i^n$ of a PU function φ_i and a local basis function ϑ_i^n. With these shape functions, we then set up a sparse linear system of equations $A\tilde{u} = \hat{f}$ via the classical Galerkin method. The linear system is then solved by our multilevel iterative solver [14, 16].

Let us now specify the particular choices for the PU functions φ_i and the local approximation space V_i employed in our PPUM. The fundamental

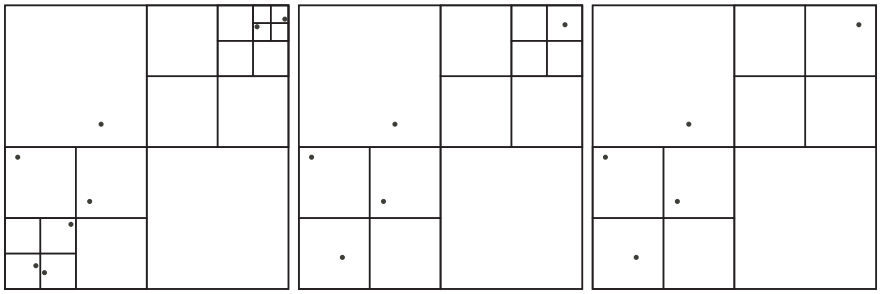

Fig. 1. Subdivision corresponding to a cover on level $J = 4$ with initial point cloud (left), derived coarser subdivisions on level 3 (center), and level 2 (right) with respective coarser point cloud.

construction principle employed in [13] for the construction of the PU $\{\varphi_i\}$ is a d-binary tree. Based on the given point data $P = \{x_i \,|\, i = 1, \ldots, \hat{N}\}$, we sub-divide a bounding-box $\mathcal{C}_\Omega \supset \Omega$ of the domain Ω until each cell

$$\mathcal{C}_i = \prod_{l=1}^{d} (c_i^l - h_i^l, c_i^l + h_i^l)$$

associated with a leaf of the tree contains at most a single point $x_i \in P$, see Figure 1. We obtain an overlapping cover $C_\Omega := \{\omega_i\}$ from this tree by defining the cover patches ω_i by simple uniform and isotropic scaling

$$\omega_i := \prod_{l=1}^{d} (c_i^l - \alpha h_i^l, c_i^l + \alpha h_i^l), \quad \text{with } \alpha > 1. \tag{2.3}$$

Note that we define a cover patch ω_i for leaf-cells $\mathcal{C}_i$ that contain a point $x_i \in P$ as well as for *empty* cells that do not contain any point from P. The coarser covers C_Ω^k are defined considering coarser versions of the constructed tree, i.e., by removing a complete set of leaves of the tree, see Figure 1. For details of this construction see [13, 14, 20].

To obtain a PU on a cover C_Ω^k with $N_k := \mathrm{card}(C_\Omega^k)$ we define a weight function $W_{i,k} : \Omega \to \mathbb{R}$ with $\mathrm{supp}(W_{i,k}) = \omega_{i,k}$ for each cover patch $\omega_{i,k}$ by

$$W_{i,k}(x) = \begin{cases} \mathcal{W} \circ T_{i,k}(x) & x \in \omega_{i,k} \\ 0 & \text{else} \end{cases} \tag{2.4}$$

with the affine transforms $T_{i,k} : \overline{\omega}_{i,k} \to [-1, 1]^d$ and $\mathcal{W} : [-1, 1]^d \to \mathbb{R}$ the reference d-linear B-spline. By simple averaging of these weight functions we obtain the functions

$$\varphi_{i,k}(x) := \frac{W_{i,k}(x)}{S_{i,k}(x)}, \quad \text{with} \quad S_{i,k}(x) := \sum_{l=1}^{N_k} W_{l,k}(x). \tag{2.5}$$

We refer to the collection $\{\varphi_{i,k}\}$ with $i = 1, \ldots, N_k$ as a partition of unity since there hold the relations

$$\begin{aligned} 0 \;\leq\; \varphi_{i,k}(x) \leq 1, \qquad & \sum_{i=1}^{N_k} \varphi_{i,k} \equiv 1 \text{ on } \overline{\Omega}, \\ \|\varphi_{i,k}\|_{L^\infty(\mathbb{R}^d)} \leq C_{\infty,k}, \qquad & \|\nabla \varphi_{i,k}\|_{L^\infty(\mathbb{R}^d)} \leq \frac{C_{\nabla,k}}{\operatorname{diam}(\omega_{i,k})} \end{aligned} \tag{2.6}$$

with constants $0 < C_{\infty,k} < 1$ and $C_{\nabla,k} > 0$ so that the assumptions of the error analysis given in [2] are satisfied by our PPUM construction. Furthermore, the PU (2.5) based on the cover $C_\Omega^k = \{\omega_{i,k}\}$ obtained from the scaling of a tree decomposition (2.3) satisfies the flat-top property (for a particular choice of $\alpha > 1$), see [16, 21].

Definition 1 (Flat-top property). *Let $\{\varphi_i\}$ be a partition of unity satisfying (2.6). Let us define the sub-patches $\omega_{\mathrm{FT,i}} \subset \omega_i$ such that $\varphi_i|_{\omega_{\mathrm{FT,i}}} \equiv 1$. Then, the PU is said to have the flat top property, if there exists a constant C_{FT} such that for all patches ω_i*

$$\mu(\omega_i) \leq C_{\mathrm{FT}}\, \mu(\omega_{\mathrm{FT,i}}) \tag{2.7}$$

where $\mu(A)$ denotes the Lebesgue measure of $A \subset \mathbb{R}^d$. We have $C_\infty = 1$ for a PU with the flat top property.

This property is essential to ensure that the product functions $\varphi_{i,k}\vartheta_{i,k}^m$ are linearly independent, provided that the employed local approximation functions $\vartheta_{i,k}^m$ are linearly independent with respect to $\omega_{\mathrm{FT,i}} = \{x \in \omega_{i,k} \,|\, \varphi_{i,k}(x) = 1\}$. Hence, we obtain global stability of the product functions $\varphi_{i,k}\vartheta_{i,k}^m$ from the local stability of the approximation functions $\vartheta_{i,k}^m$ on $\omega_{i,k}$ [20].

In general the local approximation space $V_{i,k} := \operatorname{span}\langle \vartheta_{i,k}^m \rangle$ associated with a particular patch $\omega_{i,k}$ of a PPUM space V_k^{PU} consists of two parts: A smooth approximation space, e.g. polynomials $\mathcal{P}^{p_{i,k}}(\omega_{i,k}) := \operatorname{span}\langle \psi_i^s \rangle$, and an enrichment part $\mathcal{E}_{i,k}(\omega_{i,k}) := \operatorname{span}\langle \eta_i^t \rangle$, i.e.

$$V_{i,k}(\omega_{i,k}) = \operatorname{span}\langle \vartheta_{i,k}^m \rangle = \mathcal{P}^{p_{i,k}}(\omega_{i,k}) + \mathcal{E}_{i,k}(\omega_{i,k}) = \operatorname{span}\langle \psi_i^s, \eta_i^t \rangle. \tag{2.8}$$

Note that for the smooth space $\mathcal{P}^{p_{i,k}}$ we employ a local basis $\psi_{i,k}^s$ on $\omega_{i,k}$, i.e. $\psi_{i,k}^s = p_s \circ T_{i,k}$ and $\langle p_s \rangle$ denotes a stable basis of the space of polynomials of total degree $p_{i,k}$ on $[-1, 1]^d$. The enrichment functions $\eta_{i,k}^t$ however are often given as restrictions of global functions η^t on the computational domain Ω since they are designed to capture special behavior of the solution at a particular location. Therefore, the restrictions $\langle \eta_{i,k}^t := \eta^t|_{\omega_{i,k}} \rangle$ of the enrichment functions $\langle \eta^t \rangle$ to a particular patch $\omega_{i,k}$ may be ill-conditioned or even linearly dependent on $\omega_{i,k}$, even if the enrichment functions η^t are well-conditioned on a global scale. Furthermore, the coupling between the spaces $\mathcal{P}^{p_{i,k}}$ and $\mathcal{E}_{i,k}$ on the patch $\omega_{i,k}$ must be considered since (2.8) is *not* a direct splitting. The

system of functions $\langle \vartheta_{i,k}^m \rangle = \langle \psi_{i,k}^s, \eta_{i,k}^t \rangle$ degenerates from a basis of $V_{i,k}$ to a generating system $\langle \hat{\vartheta}_{i,k}^m \rangle$ if the restricted enrichment functions $\eta_{i,k}^t = \eta^t|_{\omega_{i,k}}$ can be well approximated by polynomials $\psi_{i,k}^s$ on the patch $\omega_{i,k}$.

The elimination of the linear dependencies (i.e. the ill-conditioning) of the generating system $\langle \hat{\vartheta}_{i,k}^m \rangle$ and the selection of a well-conditioned basis $\langle \vartheta_{i,k}^m \rangle$ for the space $V_{i,k}(\omega_{i,k})$ is the main challenge in an enriched PPUM computation (and any other numerical method that employs enrichment). In [22] we have developed an appropriate projection/preconditioner $P_{i,k}^E$ that maps the generating system $\langle \hat{\vartheta}_{i,k}^m \rangle$ to such a stable basis $\langle \vartheta_{i,k}^m \rangle$. However, the separation of the smooth degrees of freedom $\psi_{i,k}^s$ and the enrichment degrees of freedom $\langle \eta_{i,k}^t \rangle$ was lost. In this paper we present an improved preconditioner $P_{i,k}$ that does not only yield a well-conditioned basis $\langle \vartheta_{i,k}^m \rangle$ for the space $V_{i,k}(\omega_{i,k})$ but also separates the enrichment degrees of freedom completely from the smooth components of $V_{i,k}(\omega_{i,k})$; i.e., we construct a direct splitting of the local function spaces $V_{i,k}$

$$V_{i,k}(\omega_{i,k}) = \operatorname{span}\langle \vartheta_{i,k}^m \rangle = \mathcal{P}^{p_{i,k}}(\omega_{i,k}) + \mathcal{E}_{i,k}(\omega_{i,k}) = \mathcal{P}^{p_{i,k}}(\omega_{i,k}) \oplus \mathcal{D}_{i,k}(\omega_{i,k})$$

in the employed smooth space $\mathcal{P}^{p_{i,k}} = \operatorname{span}\langle \psi_i^s \rangle$ and a pure enrichment space $\mathcal{D}_{i,k} := \mathcal{E}_{i,k} \setminus \mathcal{P}^{p_{i,k}} = \langle \eta_i^t \rangle$. Due to the flat-top property of the employed PU we obtain a respective stable global splitting

$$V_k^{\mathrm{PU}} = \sum_{i=1}^{N_k} \varphi_{i,k} \mathcal{P}^{p_{i,k}} \oplus \sum_{i=1}^{N_k} \varphi_{i,k} \mathcal{D}_{i,k}.$$

With the help of the shape functions $\varphi_{i,k}\vartheta_{i,k}^m$ we then discretize a PDE in weak form

$$a(u, v) = \langle f, v \rangle$$

via the classical Galerkin method to obtain a discrete linear system of equations $A\tilde{u} = \hat{f}$. Note that the PU functions (2.5) in the PPUM are in general piecewise rational functions only. Therefore, the use of an appropriate numerical integration scheme [14] is indispensable in the PPUM as in most meshfree approaches [1, 3, 8, 9]. Moreover, the functions $\varphi_{i,k}\vartheta_{i,k}^m$ in general do not satisfy the Kronecker property. Thus, the coefficients $\tilde{u}_k := (u_{i,k}^m)$ of a discrete function

$$u_k^{\mathrm{PU}} = \sum_{i=1}^{N_k} \varphi_{i,k} \sum_{m=1}^{d_{i,k}} u_{i,k}^m \vartheta_{i,k}^m = \sum_{i=1}^{N_k} \varphi_{i,k} \Big(\sum_{s=1}^{d_{i,k}^{\mathcal{P}}} u_{i,k}^s \psi_{i,k}^s + \sum_{t=1}^{d_{i,k}^{\mathcal{D}}} u_{i,k}^{t+d_{i,k}^{\mathcal{P}}} \eta_{i,k}^t \Big) \quad (2.9)$$

with $d_{i,k}^{\mathcal{P}} := \dim \mathcal{P}_{i,k}$, $d_{i,k}^{\mathcal{D}} := \dim \mathcal{D}_{i,k}$, and $d_{i,k} := \dim V_{i,k} = d_{i,k}^{\mathcal{P}} + d_{i,k}^{\mathcal{D}}$ on level k do not directly correspond to function values and a trivial interpolation of essential boundary data is not available.

2.1 Essential Boundary Conditions

The treatment of essential boundary conditions in meshfree methods is not straightforward and a number of different approaches have been suggested. In [15] we have presented how Nitsche's method [18] can be applied successfully in the meshfree context. Here, we give a short summary of this approach. To this end, let us consider the model problem

$$\begin{aligned} -\mathbf{div}\,\boldsymbol{\sigma}(u) &= f && \text{in } \Omega \subset \mathbb{R}^d \\ \boldsymbol{\sigma}(u)\cdot n &= g_N && \text{on } \Gamma_N \subset \partial\Omega \\ u\cdot n &= g_{D,n} && \text{on } \Gamma_D = \partial\Omega \setminus \Gamma_N \\ (\boldsymbol{\sigma}(u)\cdot n)\cdot t &= 0 && \text{on } \Gamma_D = \partial\Omega \setminus \Gamma_N \end{aligned} \tag{2.10}$$

In the following we drop the level subscript $k = 0, \ldots, J$ for the ease of notation.

Let us define the cover of the Dirichlet boundary

$$C_{\Gamma_D} := \{\omega_i \in C_\Omega \,|\, \Gamma_{D,i} \neq \emptyset\}$$

where $\Gamma_{D,i} := \omega_i \cap \Gamma_D$ and $\gamma_{D,i} := \operatorname{diam}(\Gamma_{D,i})$. With these conventions we define the cover-dependent functional

$$J_{C_\Omega}(w) := \int_\Omega \boldsymbol{\sigma}(w) : \boldsymbol{\epsilon}(w)\; dx - 2\int_{\Gamma_D} (n\cdot\boldsymbol{\sigma}(w)n)n\cdot w\; ds + \beta \sum_{\omega_i \in C_{\Gamma_D}} \gamma_{D,i}^{-1} \int_{\Gamma_{D,i}} (w\cdot n)^2\; ds$$

with some parameter $\beta > 0$. Minimizing J_{C_Ω} with respect to the error $u - u^{\mathrm{PU}}$ yields the weak formulation

$$a_{C_\Omega}(w, v) = l_{C_\Omega}(v) \quad \text{for all } v \in V^{\mathrm{PU}} \tag{2.11}$$

with the cover-dependent bilinear form

$$\begin{aligned} a_{C_\Omega}(u, v) := &\int_\Omega \boldsymbol{\sigma}(u) : \boldsymbol{\epsilon}(v)\; dx - \int_{\Gamma_D} (n\cdot\boldsymbol{\sigma}(u)n)n\cdot v\; ds \\ &- \int_{\Gamma_D} (n\cdot\boldsymbol{\sigma}(v)n)n\cdot u\; ds + \beta \sum_{\omega_i \in C_{\Gamma_D}} \gamma_{D,i}^{-1} \int_{\Gamma_{D,i}} (u\cdot n)(v\cdot n)\; ds \end{aligned}$$

and the corresponding linear form

$$\langle l_{C_\Omega}, v\rangle := \int_\Omega fv + \int_{\Gamma_N} g_N v - \int_{\Gamma_D} g_{D,n}(n\cdot\boldsymbol{\sigma}(v)n) + \beta \sum_{\omega_i \in C_{\Gamma_D}} \gamma_{D,i}^{-1} \int_{\Gamma_{D,i}} g_{D,n}(v\cdot n)\; ds$$

There is a unique solution u^{PU} of (2.11) if the regularization parameter β is chosen large enough; i.e., the regularization parameter $\beta = \beta_{V^{\mathrm{PU}}}$ is dependent on the discretization space V^{PU}. This solution u^{PU} satisfies optimal error bounds if the space V^{PU} admits the inverse estimate

$$\|(n \cdot \boldsymbol{\sigma}(v)n)\|^2_{-\frac{1}{2},C_{\Gamma_D}} \;\leq\; C^2_{V^{\mathrm{PU}}}\|v\|^2_E \;=\; C^2_{V^{\mathrm{PU}}}\int_\Omega \boldsymbol{\sigma}(v) : \boldsymbol{\epsilon}(v)\; dx \qquad (2.12)$$

for all $v \in V^{\mathrm{PU}}$ with respect to the cover-dependent norm

$$\|w\|^2_{-\frac{1}{2},C_{\Gamma_D}} := \sum_{\omega_i \in C_{\Gamma_D}} \gamma_{D,i}\|w\|^2_{L^2(\Gamma_{D,i})}$$

with a constant $C_{V^{\mathrm{PU}}}$ depending on the cover C_Ω and the employed local bases $\langle \vartheta^n_i \rangle$ only. If $C_{V^{\mathrm{PU}}}$ is known, the regularization parameter $\beta_{V^{\mathrm{PU}}}$ can be chosen as $\beta_{V^{\mathrm{PU}}} > 2C^2_{V^{\mathrm{PU}}}$ to obtain a symmetric positive definite linear system [18]. Hence, the main task associated with the use of Nitsche's approach in the PPUM context is the efficient and automatic computation of the constant $C_{V^{\mathrm{PU}}}$, see [15,20]. To this end, we consider the inverse assumption (2.12) as a generalized eigenvalue problem locally on each patch $\omega_i \in C_{\Gamma_D}$ and solve for the largest eigenvalue to obtain an approximation of $C^2_{V^{\mathrm{PU}}}$.

In summary, the PPUM discretization of our model problem (2.10) using the space V^{PU} on the cover C_Ω is carried out in two steps: First, we estimate the regularization parameter $\beta_{V^{\mathrm{PU}}}$ from (2.12). Then, we define the weak form (2.11) and use Galerkin's method to set up the respective symmetric positive definite linear system $A\tilde{u} = \hat{f}$. This linear system is then solved by our multilevel iterative solver [14,16].

3 Enrichment and Local Preconditioning

The use of smooth polynomial local approximation spaces $V_{i,k} = \mathcal{P}^{p_{i,k}}$ in our PPUM is optimal only for the approximation of a smooth or regular solution u. In the case of a discontinuous or singular solution u there are essentially two approaches we can pursue: First, a classical adaptive refinement process which essentially resolves the singular behavior of the solution by geometric subdivision, see [16,21]. Second, an algebraic approach that is very natural to the PPUM, the explicit enrichment of the global approximation space by special shape functions η^s. This approach is also pursued in other meshfree methods [5,19], the XFEM [4,6,17] or the GFEM [10–12]. Most enrichment schemes however focus on modeling issues and not on approximation properties or the conditioning of the resulting stiffness matrix.

In [22] we presented an hierarchical enrichment scheme which recovers the optimal convergence rate of the uniform h-version independently of the regularity of the solution. Let us shortly summarize this construction here. To this end, we consider the reference problem

$$\begin{aligned} -\mathbf{div}\,\boldsymbol{\sigma}(u) &= f \quad \text{in } \Omega = (-1,1)^2, \\ \boldsymbol{\sigma}(u)\cdot n &= g_N \text{ on } \Gamma_N \subset \partial\Omega \cup C, \\ u &= g_D \text{ on } \Gamma_D = \partial\Omega \setminus \Gamma_N, \end{aligned} \qquad (3.1)$$

from linear elastic fracture mechanics where the internal traction-free segment

$$C := \{(x, y) \in \Omega \,|\, x \in (-0.5, 0.5) \text{ and } y = 0\}$$

is referred to as a crack. The crack C induces a discontinuous displacement field u across the crack line C with singularities at the crack tips $c_l := (-0.5, 0)$ and $c_u := (0.5, 0)$. Hence, the local approximation spaces $V_{i,k}$ employed in our PPUM must respect these features to provide good approximation.

Thus on patches $\omega_{i,k}$ with

$$\omega_{i,k} \cap C \neq \emptyset \quad \text{and} \quad \{c_l, c_u\} \cap \omega_{i,k} = \emptyset$$

we enrich the smooth local approximation space $\mathcal{P}^{p_{i,k}}$ by

$$\mathcal{E}_{i,k} := H_{\pm}^{C} \mathcal{P}^{p_{i,k}} \quad \text{so that} \quad V_{i,k} := \mathcal{P}^{p_{i,k}} + H_{\pm}^{C} \mathcal{P}^{p_{i,k}} \tag{3.2}$$

where $H_{\pm}^{C}$ denotes the Haar function that is discontinuous at the crack C. If a patch $\omega_{i,k}$ contains a crack tip ξ_{tip}, i.e. $c_l \in \omega_{i,k}$ or $c_u \in \omega_{i,k}$, then the patch is enriched by the respective space of singular tip functions

$$W_{\text{tip}} := \{\sqrt{r}\cos\frac{\theta}{2}, \sqrt{r}\sin\frac{\theta}{2}, \sqrt{r}\sin\theta\sin\frac{\theta}{2}, \sqrt{r}\sin\theta\cos\frac{\theta}{2}\} \tag{3.3}$$

given in local polar coordinates with respect to the tip ξ_{tip}, i.e. $\mathcal{E}_{i,k} = W_{\text{tip}}|_{\omega_{i,k}}$. This yields the local approximation space

$$V_{i,k} := \mathcal{P}^{p_{i,k}} + W_{\text{tip}}$$

for a patch $\omega_{i,k}$ that contains the tip ξ_{tip}.

So far this is standard and employed (in a similar way)[1] in many numerical methods [4–6, 10–12, 17, 19, 21]. However, the above enrichment scheme does not yield an improvement of the asymptotic convergence behavior since the singular enrichment is employed only directly at the tip. Yet with the help of our multilevel sequence of covers C_{Ω}^{k} we can easily generalize this enrichment scheme to obtain enriched patches in an appropriate vicinity of the tips of C.

Instead of enriching only those patches $\omega_{i,k}$ by (3.3) which satisfy $\{c_l, c_u\} \cap \omega_{i,k} \neq \emptyset$ we enrich all patches $\omega_{i,k}$ that satisfy

$$\omega_{i,k} \subset \omega_{\tilde{i},k-1} \quad \text{with} \quad \mathcal{E}_{\tilde{i},k-1} = W_{\text{tip}}$$

for a particular coarser patch $\omega_{\tilde{i},k-1} \in C_{\Omega}^{k-1}$.[2] Observe that with this approach we obtain enrichment spaces

$$\mathcal{E}_{i,k} := W_{\text{tip}} + H_{\pm}^{C} \mathcal{P}^{p_{i,k}} \quad \text{so that} \quad V_{i,k} := \mathcal{P}^{p_{i,k}} + W_{\text{tip}} + H_{\pm}^{C} \mathcal{P}^{p_{i,k}} \tag{3.4}$$

for patches $\omega_{i,k}$ satisfying

[1] Note that we use a fully multiplicative enrichment in (3.2) whereas most other enrichment schemes employ just an additive enrichment with $\mathcal{E} = \text{span}\langle H_{\pm}^{C} \rangle$.

[2] Due to the employed tree-construction $\omega_{\tilde{i},k-1} \in C_{\Omega}^{k-1}$ with $\omega_{i,k} \subset \omega_{\tilde{i},k-1}$ is unique for an arbitrary patch $\omega_{i,k} \in C_{\Omega}^{k}$.

$$\omega_{i,k} \cap C \neq \emptyset, \quad \{c_l, c_u\} \cap \omega_{i,k} = \emptyset, \quad \text{and} \quad \omega_{i,k} \subset \omega_{\tilde{i},k-1} \text{ with } \mathcal{E}_{\tilde{i},k-1} = W_{\text{tip}}.$$

This recursive enrichment process yields a constant enrichment zone

$$E := \bigcap_{k=0}^{J\to\infty} E_k \quad \text{where} \quad E_k := \bigcup_{\substack{i=1 \\ \mathcal{E}_{i,k} \supset W_{\text{tip}}}}^{N_k} \omega_{i,k}. \tag{3.5}$$

Hence, the singularities of the solution u are not only resolved by a constant number of patches $\omega_{i,k}$ on each level k but by an increasing number of patches on each level k. The singularities of the solution are resolved in the enrichment zone (3.5) on all levels k. Thus, our PPUM will converge with a rate that is independent of the regularity of the solution (i.e. regularity with respect to enrichment zone E). However, this approach amplifies the ill-conditioning of the stiffness matrix since a large number of enrichment functions away from the singularities of u is employed.

There are two sources of ill-conditioning in an enriched PPUM using the local approximation spaces $V_{i,k} = \mathcal{P}^{p_{i,k}} + \mathcal{E}_{i,k}$ with $\mathcal{P}^{p_{i,k}} = \text{span}\langle \psi^s_{i,k} \rangle$ and $\mathcal{E}_{i,k} = \text{span}\langle \eta^t_{i,k} \rangle$. First of all, the restrictions $\eta^t_{i,k} := \eta^t|_{\omega_{i,k}}$ of the global enrichment functions η^t to the local patches $\omega_{i,k}$ may be (almost) linearly dependent *locally* on $\omega_{i,k} \cap \Omega$ even though the global enrichment functions η^t are linearly independent (well-conditioned) *globally* on Ω. Furthermore, on patches $\omega_{i,k} \cap C = \emptyset$ the restrictions $\eta^t_{i,k}$ are regular functions and thus some enrichment functions $e_{i,k} \in \mathcal{E}_{i,k}$ may be approximated well by local polynomials $\psi^s_{i,k}$; i.e. $e_{i,k} \in \mathcal{P}^{p_{i,k}}$. Hence, the condition number associated with the system of functions $\langle \psi^s_{i,k}, \eta^t_{i,k} \rangle$ can deteriorate very rapidly.

In the following we present an efficient algebraic approach to overcome the ill-conditioning of local basis $\langle \psi^s_{i,k}, \eta^t_{i,k} \rangle$ (and ultimately of the resulting stiffness matrix) due to enrichment where we drop the level subscript k for the ease of notation. The overall goal of our construction is to attain a direct splitting of a local approximation space

$$V_i = \mathcal{P}^{p_i} + \mathcal{E}_i = \mathcal{P}^{p_i} \oplus \mathcal{D}_i$$

into the smooth sub-space $\mathcal{P}^{p_i}$ and a pure enrichment space $\mathcal{D}_i := \mathcal{E}_i \setminus \mathcal{P}^{p_i}$. To this end let us introduce some short-hand notation and the spaces

$$\begin{aligned}
\mathcal{P}^{p_i} &:= \text{span}\langle \psi^s_i \rangle, \text{ with } d^{\mathcal{P}}_i := \dim(\mathcal{P}^{p_i}) = \text{card}(\{\psi^s_i\}), \\
\mathcal{E}_i &:= \text{span}\langle \hat{\eta}^t_i \rangle \,, \text{ with } d^{\mathcal{E}}_i := \dim(\mathcal{E}_i) \quad \leq \text{card}(\{\hat{\eta}^t_i\}) \quad =: c^{\mathcal{E}}_i, \\
V_i &:= \mathcal{P}^{p_i} + \mathcal{E}_i \,, \text{ with } d_i := \dim(V_i) \quad \leq d^{\mathcal{P}}_i + c^{\mathcal{E}}_i \qquad =: c_i;
\end{aligned} \tag{3.6}$$

i.e., we assume that we have a stable basis $\langle \psi^s_i \rangle$ for $\mathcal{P}^{p_i}$ and only a generating system $\langle \hat{\eta}^t_i \rangle$ for $\mathcal{E}_i$. Thus, we also have a generating system $\langle \hat{\vartheta}^m_i \rangle := \langle \psi^s_i, \hat{\eta}^t_i \rangle$ for the local approximation space V_i on the respective patch ω_i only.

With respect to the generating system $\langle \hat{\vartheta}_i^m \rangle = \langle \psi_i^s, \hat{\eta}_i^t \rangle$ we obtain the singular (ill-conditioned) local mass matrix M_i with entries $(M_i)_{n,m}$ on a particular patch ω_i by

$$(M_i)_{n,m} := \int_{\omega_i \cap \Omega} \hat{\vartheta}_i^n \hat{\vartheta}_i^m \, dx \quad \text{for all } m, n = 1, \dots, c_i \tag{3.7}$$

which can be written in the block-form

$$M_i = \begin{pmatrix} M_{\mathcal{P},\mathcal{P}}^i & M_{\mathcal{P},\mathcal{E}}^i \\ M_{\mathcal{E},\mathcal{P}}^i & M_{\mathcal{E},\mathcal{E}}^i \end{pmatrix}$$

due to the additive representation $V_i = \mathcal{P}^{p_i} + \mathcal{E}_i$. For the ease of notation we furthermore drop the sub- and superscript i in the following; i.e., we consider the local mass matrix M on a particular patch ω_i in block-form

$$M := \begin{pmatrix} M_{\mathcal{P},\mathcal{P}} & M_{\mathcal{P},\mathcal{E}} \\ M_{\mathcal{E},\mathcal{P}} & M_{\mathcal{E},\mathcal{E}} \end{pmatrix} \tag{3.8}$$

where the blocks satisfy

$$M_{\mathcal{P},\mathcal{P}} \in \mathbb{R}^{d_i^{\mathcal{P}} \times d_i^{\mathcal{P}}}, \quad M_{\mathcal{E},\mathcal{E}} \in \mathbb{R}^{c_i^{\mathcal{E}} \times c_i^{\mathcal{E}}}, \quad M_{\mathcal{E},\mathcal{P}} = M_{\mathcal{P},\mathcal{E}}^T \in \mathbb{R}^{c_i^{\mathcal{E}} \times d_i^{\mathcal{P}}}. \tag{3.9}$$

Note that the linear dependencies among the enrichment functions $\hat{\eta}_i^t$ lead to a deterioration of the condition number of the block $M_{\mathcal{E},\mathcal{E}}$; i.e., the matrix $M_{\mathcal{E},\mathcal{E}}$ has a non-zero (numerical) kernel or (near-)null space. The elements of the (near-)null space are not necessary to construct any element of the function space $\mathcal{E}_i$ and hence can be eliminated. Thus, we look for a system of functions $\langle \tilde{\eta}_i^t \rangle$ that spans the space $\mathcal{E}_i$ and at the same time allows for a simple elimination of the kernel elements of $M_{\mathcal{E},\mathcal{E}}$. The eigenfunctions (eigenvectors) of $M_{\mathcal{E},\mathcal{E}}$ provide such a system $\langle \tilde{\eta}_i^t \rangle$. Hence, let us consider the eigenvalue decomposition

$$O_{\mathcal{E}}^T M_{\mathcal{E},\mathcal{E}} O_{\mathcal{E}} = D_{\mathcal{E}} \quad \text{with } O_{\mathcal{E}}, D_{\mathcal{E}} \in \mathbb{R}^{c_i^{\mathcal{E}} \times c_i^{\mathcal{E}}} \tag{3.10}$$

of the matrix $M_{\mathcal{E},\mathcal{E}}$ with the normal transformation $O_{\mathcal{E}}^T$ and the diagonal matrix $D_{\mathcal{E}}$, i.e.,

$$O_{\mathcal{E}}^T O_{\mathcal{E}} = \mathbb{I}_{c_i^{\mathcal{E}}}, \quad \text{and } (D_{\mathcal{E}})_{r,q} = 0 \quad \text{for all } r, q = 1, \dots, c_i^{\mathcal{E}}, \text{ and } r \neq q.$$

From (3.10) we can easily identify the (numerical) kernel of $M_{\mathcal{E},\mathcal{E}}$ if we assume that the eigenvalues $(D_{\mathcal{E}})_{r,r}$ are given in decreasing order, i.e. $(D_{\mathcal{E}})_{m,m} \geq (D_{\mathcal{E}})_{m+1,m+1}$. Then the matrices $O_{\mathcal{E}}^T$ and $D_{\mathcal{E}}$ are partitioned as

$$O_{\mathcal{E}}^T = \begin{pmatrix} \tilde{O}_{\mathcal{E}}^T \\ K_{\mathcal{E}}^T \end{pmatrix}, \quad \text{and} \quad D_{\mathcal{E}} = \begin{pmatrix} \tilde{D}_{\mathcal{E}} & 0 \\ 0 & \kappa_{\mathcal{E}} \end{pmatrix} \tag{3.11}$$

where the rth row of the rectangular matrix $\tilde{O}_{\mathcal{E}}^T \in \mathbb{R}^{d_i^{\mathcal{E}} \times c_i^{\mathcal{E}}}$ is an eigenvector of $M_{\mathcal{E},\mathcal{E}}$ that is associated with an eigenvalue $(\tilde{D}_{\mathcal{E}})_{r,r}$ satisfying $(D_{\mathcal{E}})_{r,r} =$

$(\tilde{D}_{\mathcal{E}})_{r,r} \geq \epsilon\ (\tilde{D}_{\mathcal{E}})_{0,0} = (D_{\mathcal{E}})_{0,0}$ and $K_{\mathcal{E}}^T$ involves all eigenvectors that are associated with small eigenvalues, i.e. $(\kappa_{\mathcal{E}})_{q,q} < \epsilon\ (\tilde{D}_{\mathcal{E}})_{0,0}$.[3] Thus, the inverse of $\tilde{D}_{\mathcal{E}}$ can be computed stably and the operator

$$\tilde{D}_{\mathcal{E}}^{-1/2}\tilde{O}_{\mathcal{E}}^T : \mathcal{E} = \text{span}\langle\hat{\eta}_i^t\rangle \to \mathcal{E} = \text{span}\langle\tilde{\eta}_i^t\rangle$$

gives a stable basis $\langle\tilde{\eta}_i^t\rangle$ of the enrichment space $\mathcal{E}_i$ and there holds $d_i^{\mathcal{E}} = \dim(\mathcal{E}_i) = \text{card}(\{\tilde{\eta}_i^t\}) \leq c_i^{\mathcal{E}} = \text{card}(\{\hat{\eta}_i^t\})$. With respect to the basis $\langle\tilde{\eta}_i^t\rangle$ the block $M_{\mathcal{E},\mathcal{E}}$ of the local mass matrix M is given as

$$M_{\mathcal{E},\mathcal{E}}^* := \tilde{D}_{\mathcal{E}}^{-1/2}\tilde{O}_{\mathcal{E}}^T M_{\mathcal{E},\mathcal{E}}\tilde{O}_{\mathcal{E}}\tilde{D}_{\mathcal{E}}^{-1/2} = \mathbb{I}_{d_i^{\mathcal{E}}};$$

i.e., the operator $\tilde{D}_{\mathcal{E}}^{-1/2}\tilde{O}_{\mathcal{E}}^T$ yields an optimally conditioned orthonormal basis $\langle\tilde{\eta}_i^t\rangle$ for the enrichment space $\mathcal{E}_i$ on the patch ω_i. Applying the block-transformation

$$T_{\mathcal{E}}^T := \begin{pmatrix} \mathbb{I}_{d^{\mathcal{P}_i}} & 0 \\ 0 & \tilde{D}_{\mathcal{E}}^{-1/2}\tilde{O}_{\mathcal{E}}^T \end{pmatrix}$$

to the local mass matrix M of (3.8) we obtain

$$M_{\mathcal{E}} := T_{\mathcal{E}}^T M T_{\mathcal{E}} = \begin{pmatrix} M_{\mathcal{P},\mathcal{P}} & M_{\mathcal{P},\mathcal{E}}^* \\ M_{\mathcal{E},\mathcal{P}}^* & M_{\mathcal{E},\mathcal{E}}^* \end{pmatrix}$$

the local mass matrix with respect to the collection of functions $\langle\psi_i^s, \tilde{\eta}_i^t\rangle$ with the blocks

$$M_{\mathcal{P},\mathcal{E}}^* := (M_{\mathcal{E},\mathcal{P}}^*)^T, \qquad M_{\mathcal{E},\mathcal{P}}^* := \tilde{D}_{\mathcal{E}}^{-1/2}\tilde{O}_{\mathcal{E}}^T M_{\mathcal{E},\mathcal{P}}, \quad \text{and}$$

$$M_{\mathcal{E},\mathcal{E}}^* := \tilde{D}_{\mathcal{E}}^{-1/2}\tilde{O}_{\mathcal{E}}^T M_{\mathcal{E},\mathcal{E}}\tilde{O}_{\mathcal{E}}\tilde{D}_{\mathcal{E}}^{-1/2} = \mathbb{I}_{d_i^{\mathcal{E}}}.$$

Even though both bases $\langle\psi_i^s\rangle$ of $\mathcal{P}^{p_i}$ and $\langle\tilde{\eta}_i^t\rangle$ of $\mathcal{E}_i$ are stable their simple merger $\langle\psi_i^s, \tilde{\eta}_i^t\rangle$ may not be stable. That is we currently have a representation of the local approximation space

$$V_i = \mathcal{P}^{p_i} + \mathcal{E}_i = \text{span}\langle\psi_i^s\rangle + \text{span}\langle\tilde{\eta}_i^t\rangle$$

which is *not* a direct splitting; i.e., the spaces $\mathcal{P}^{p_i}$ and $\mathcal{E}_i$ overlap in general. In the following we construct a stable direct splitting of V_i where we separate the enrichment degrees of freedom from the polynomials completely; i.e., we derive the direct splitting

$$V_i = \mathcal{P}^{p_i} \oplus \mathcal{D}_i := \mathcal{P}^{p_i} \oplus \left(\mathcal{E}_i \setminus \mathcal{P}^{p_i}\right)$$

[3] In our implementation we use the relative condition $(\tilde{D}_{\mathcal{E}})_{r,r} \geq \epsilon\ (\tilde{D}_{\mathcal{E}})_{0,0}$ and the absolute condition $(\tilde{D}_{\mathcal{E}})_{0,0} \geq \epsilon$.

and compute an appropriate basis η_i for the space $\mathcal{D}_i := \mathcal{E}_i \backslash \mathcal{P}^{p_i}$ automatically. That is we eliminate all polynomial components in the enrichment space.[4] This is achieved by a block-elimination approach; i.e. the Schur complement.

To this end, we consider the local mass matrix in block-form

$$M_{\mathcal{E}} = T_{\mathcal{E}}^T M T_{\mathcal{E}} = \begin{pmatrix} M_{\mathcal{P},\mathcal{P}} & M_{\mathcal{P},\mathcal{E}}^* \\ M_{\mathcal{E},\mathcal{P}}^* & M_{\mathcal{E},\mathcal{E}}^* \end{pmatrix}.$$

Since we assume that the basis of $\mathcal{P}^{p_i}$ is stable we can compute $(M_{\mathcal{P},\mathcal{P}})^{-1}$. Therefore the transformation

$$T_S^T := \begin{pmatrix} \mathbb{I}_{d_i^{\mathcal{P}}} & 0 \\ -M_{\mathcal{E},\mathcal{P}}^* (M_{\mathcal{P},\mathcal{P}})^{-1} & \mathbb{I}_{d_i^{\mathcal{E}}} \end{pmatrix}$$

from the system $\langle \psi_i^s, \tilde{\eta}_i^t \rangle$ to $\langle \psi_i^s, \bar{\eta}_i^t \rangle$ is stable and we obtain the block-diagonal local mass matrix

$$M_S := T_S^T M_{\mathcal{E}} T_S = \begin{pmatrix} M_{\mathcal{P},\mathcal{P}} & 0 \\ 0 & M_{\mathcal{D},\mathcal{D}} \end{pmatrix}$$

with

$$M_{\mathcal{D},\mathcal{D}} := M_{\mathcal{E},\mathcal{E}}^* - M_{\mathcal{E},\mathcal{P}}^* (M_{\mathcal{P},\mathcal{P}})^{-1} M_{\mathcal{P},\mathcal{E}}^*.$$

The matrix $M_{\mathcal{D},\mathcal{D}}$, i.e. the Schur complement matrix, is of dimension $d_i^{\mathcal{E}} \times d_i^{\mathcal{E}}$ however the dimension $d_i^{\mathcal{D}} := \dim(\mathcal{D}_i)$ of the space $\mathcal{D}_i = \mathcal{E}_i \setminus \mathcal{P}^{p_i} = \operatorname{span}\langle \bar{\eta}_i^t \rangle$ may be smaller. This is the case if a non-vanishing linear combination of the local restrictions η_i^t of the global enrichment functions η_t can be represented (or well-approximated) by polynomials. Thus, $M_{\mathcal{D},\mathcal{D}}$ in general has a (numerical) kernel and we need to eliminate these kernel elements to obtain a stable local mass matrix on ω_i. To this end, we employ the same approach as for the block $M_{\mathcal{E},\mathcal{E}}$. We compute the eigenvalue decomposition

$$O_{\mathcal{D}}^T M_{\mathcal{D},\mathcal{D}} O_{\mathcal{D}} = D_{\mathcal{D}} \quad \text{with } O_{\mathcal{D}}, D_{\mathcal{D}} \in \mathbb{R}^{d_i^{\mathcal{E}} \times d_i^{\mathcal{E}}} \tag{3.12}$$

of the Schur complement matrix $M_{\mathcal{D},\mathcal{D}}$ where

$$O_{\mathcal{D}}^T O_{\mathcal{D}} = \mathbb{I}_{d_i^{\mathcal{E}}}, \qquad (D_{\mathcal{D}})_{r,q} = 0 \quad \text{for all } r, q = 1, \dots, d_i^{\mathcal{E}}, \text{ and } r \neq q.$$

Again, we can easily identify the (numerical) kernel in the eigenbasis. Analogously to (3.11) we partition the matrices $O_{\mathcal{D}}^T$ and $D_{\mathcal{D}}$ of (3.12) as

$$O_{\mathcal{D}}^T = \begin{pmatrix} \tilde{O}_{\mathcal{D}}^T \\ K_{\mathcal{D}}^T \end{pmatrix}, \quad \text{and} \quad D_{\mathcal{D}} = \begin{pmatrix} \tilde{D}_{\mathcal{D}} & 0 \\ 0 & \kappa_{\mathcal{D}} \end{pmatrix} \tag{3.13}$$

with $\tilde{D}_{\mathcal{D}} \in \mathbb{R}^{d_i^{\mathcal{D}} \times d_i^{\mathcal{D}}}$ is invertible with maximal $d_i^{\mathcal{D}}$. Thus, we obtain the respective transformation to the eigenbasis $\langle \eta_i^t \rangle$ of the Schur complement $M_{\mathcal{D},\mathcal{D}}$ by

[4] Since the matrix $M_{\mathcal{E},\mathcal{E}}^* = \mathbb{I}_{d_i^{\mathcal{E}}}$ is invertible due to our construction we can also eliminate the enrichment functions from the polynomials.

$$\tilde{D}_{\mathcal{D}}^{-1/2}\tilde{O}_{\mathcal{D}}^{T} : \mathcal{D}_i = \operatorname{span}\langle \bar{\eta}_i^t \rangle \to \mathcal{D}_i = \operatorname{span}\langle \eta_i^t \rangle.$$

Hence, the block-diagonal transformation

$$T_{\mathcal{D}}^{T} := \begin{pmatrix} \mathbb{I}_{d_i^{\mathcal{P}}} & 0 \\ 0 & \tilde{D}_{\mathcal{D}}^{-1/2}\tilde{O}_{\mathcal{D}}^{T} \end{pmatrix}$$

yields a symmetric positive definite local mass matrix $M_{\mathcal{D}}$

$$M_{\mathcal{D}} := T_{\mathcal{D}}^{T} M_S T_{\mathcal{D}} = \begin{pmatrix} M_{\mathcal{P},\mathcal{P}} & 0 \\ 0 & M_{\mathcal{D},\mathcal{D}}^{*} \end{pmatrix},$$

with $M_{\mathcal{D},\mathcal{D}}^{*} = \mathbb{I}_{d_i^{\mathcal{D}}}$; i.e., the system of functions $\langle \psi_i^s, \eta_i^t \rangle$ is a stable and well-conditioned basis of the local approximation space $V_i = \mathcal{P}^{p_i} + \mathcal{E}_i = \mathcal{P}^{p_i} \oplus \mathcal{D}_i$. The condition number of $M_{\mathcal{D}}$ is given by the condition number of $M_{\mathcal{P},\mathcal{P}}$. There is no ill-conditioning due to enrichment and the presented construction yields an optimal local preconditioner. Moreover, there holds the representation

$$V_i = \mathcal{P}^{p_i} \oplus \mathcal{D}_i$$

as a direct sum, where the space $\mathcal{D}_i = \mathcal{E}_i \setminus \mathcal{P}^{p_i}$ consists of pure enrichment degrees of freedom that cannot be represented (or well-approximated) by polynomials.

The final local preconditioner $P_i^T : V_i \to V_i$ which maps the original ill-conditioned local generating system $\langle \hat{\vartheta}_i^m \rangle = \langle \psi_i^s, \hat{\eta}_i^t \rangle$ to an optimally conditioned local basis $\langle \vartheta_i^m \rangle = \langle \psi_i^s, \eta_i^t \rangle$ of V_i is given by the product of the constructed transformations and projections

$$P_i^T := T_{\mathcal{D}}^{T} \circ T_S^{T} \circ T_{\mathcal{E}}^{T}.$$

Due to the use of a flat-top PU in our PPUM this local stability of $\langle \vartheta_i^m \rangle$ already yields stability of the global basis $\langle \varphi_i \vartheta_i^m \rangle$ of V^{PU}.[5]

Remark 1. Note that in the presentation above the equivalencies

$$V_i = \mathcal{P}^{p_i} \oplus \mathcal{D}_i = \mathcal{P}^{p_i} + \mathcal{E}_i$$

hold only up to the employed numerical cut-off parameter $\epsilon > 0$; i.e. we have

$$V_i := \mathcal{P}^{p_i} \oplus \mathcal{D}_i \approx \mathcal{P}^{p_i} + \mathcal{E}_i \quad \text{and} \quad \mathcal{D}_i \approx \mathcal{E}_i \setminus \mathcal{P}^{p_i}.$$

Remark 2. Even though we considered the identity operator $\mathbb{I}$ on the local patch ω_i, i.e. the local mass matrix M_i, we can construct the respective preconditioner for an arbitrary (definite) operator. A change of this operator impacts the absolute value of the constant condition number associated with the constructed basis only—in exact arithmetic and with $\epsilon = 0$. However, due

[5] Formally, we need to compute the entries (3.7) with respect to $\omega_{\mathrm{FT},i}$.

to the employed cut-off parameter $\epsilon > 0$ we may obtain a (slightly) different space $\operatorname{span}\langle\vartheta_{i,k}^m\rangle$ for different operators with the same ϵ. Since we are usually interested in the approximation of $u \in H^1$ we use the H^1-norm to construct the preconditioner; i.e., we employ the operator $-\Delta + \mathbb{I}$ for the construction of the basis $\langle\vartheta_i^m\rangle = \langle\psi_i^s, \eta_i^t\rangle$ of V_i.

Remark 3. Observe that we do *not* need to assemble the stiffness matrix $A = A_\vartheta$ directly with respect to the computed basis $\langle\vartheta_i^m\rangle$ (which might lead to a fair amount of implementation work and requires the application of the *dense* local preconditioners P_i^T for each evaluation/quadrature point). We can rather carry out the assembly of the stiffness matrix $A_{\hat{\vartheta}}$ using the original generating system $\langle\hat{\vartheta}_i^m\rangle$ and apply the block-diagonal preconditioner P^T with the entries

$$(P^T)_{i,j} := \begin{cases} P_i^T & j = i, \\ 0 & j \neq i, \end{cases} \tag{3.14}$$

after the assembly of $A_{\hat{\vartheta}}$. That is we attain the stiffness matrix A_ϑ with respect to the computed basis $\langle\vartheta_i^m\rangle$ as the triple-product[6]

$$A_\vartheta := P^T A_{\hat{\vartheta}} P \tag{3.15}$$

via a simple post-processing operation.

Let us summarize the presented local preconditioning approach on a particular patch $\omega_i \in C_\Omega$ by the following algorithm. Here, we include also a transformation for the space $\mathcal{P}^{p_i} = \operatorname{span}\langle\psi_i^s\rangle$ to obtain an orthonormal polynomial basis $\langle\psi_i^s\rangle$. Thus, we use Algorithm 1 for all patches $\omega_{i,k}$ not only the enriched patches.

Algorithm 1 (Local enrichment preconditioning).

1. Assemble the local matrix M on ω_i using the generating system $\langle\hat{\vartheta}_i^m\rangle = \langle\psi_i^s, \hat{\eta}_i^t\rangle$ analogously to (3.7) using the weak formulation of the considered operator, see Remark 2. Define the respective sub-matrices $M_{\mathcal{P},\mathcal{P}}$, $M_{\mathcal{E},\mathcal{E}}$, $M_{\mathcal{E},\mathcal{P}}$ of (3.9) due to the partitioning (3.8).
2. Compute the eigenvalue decomposition (3.10) of the block $M_{\mathcal{E},\mathcal{E}}$ and define the respective sub-matrices $\tilde{O}_{\mathcal{E}}^T$ and $\tilde{D}_{\mathcal{E}}$ according to (3.11) such that $\tilde{D}_{\mathcal{E}}$ is invertible.
3. Compute the eigenvalue decomposition

$$O_{\mathcal{P}}^T M_{\mathcal{P},\mathcal{P}} O_{\mathcal{P}} = D_{\mathcal{P}} \quad \text{with } O_{\mathcal{P}}, D_{\mathcal{P}} \in \mathbb{R}^{d_i^{\mathcal{P}} \times d_i^{\mathcal{P}}} \tag{3.16}$$

of the block $M_{\mathcal{P},\mathcal{P}}$ with the normal transformation $O_{\mathcal{P}}^T$ and the invertible diagonal matrix $D_{\mathcal{P}}$.

[6]Note however since P^T is block-diagonal this operation is easily parallelizable.

4. Define the block-diagonal transformation

$$T_B^T := \begin{pmatrix} D_{\mathcal{P}}^{-1/2} O_{\mathcal{P}}^T & 0 \\ 0 & \tilde{D}_{\mathcal{E}}^{-1/2} \tilde{O}_{\mathcal{E}}^T \end{pmatrix}$$

from $V_i = \operatorname{span}\langle \hat{\psi}_i^s, \hat{\eta}_i^t \rangle$ to $V_i = \operatorname{span}\langle \psi_i^s, \tilde{\eta}_i^t \rangle$.

5. Define the sub-matrix

$$M_{\mathcal{E},\mathcal{P}}^* := \tilde{D}_{\mathcal{E}}^{-1/2} \tilde{O}_{\mathcal{E}}^T M_{\mathcal{E},\mathcal{P}} O_{\mathcal{P}} D_{\mathcal{P}}^{-1/2} \tag{3.17}$$

of the transformed local matrix $T_B^T M T_B$.

6. Define the block-triangular transformation

$$T_S^T := \begin{pmatrix} \mathbb{I}_{d_i^{\mathcal{P}}} & 0 \\ -M_{\mathcal{E},\mathcal{P}}^* & \mathbb{I}_{d_i^{\mathcal{E}}} \end{pmatrix}$$

from $V_i = \operatorname{span}\langle \psi_i^s, \tilde{\eta}_i^t \rangle$ to $V_i = \operatorname{span}\langle \psi_i^s, \bar{\eta}_i^t \rangle$.

7. Define the Schur complement matrix

$$\begin{aligned} M_{\mathcal{D},\mathcal{D}} &:= \mathbb{I}_{d_i^{\mathcal{E}}} - M_{\mathcal{E},\mathcal{P}}^* M_{\mathcal{P},\mathcal{E}}^* \\ &= \mathbb{I}_{d_i^{\mathcal{E}}} - \tilde{D}_{\mathcal{E}}^{-1/2} \tilde{O}_{\mathcal{E}}^T M_{\mathcal{E},\mathcal{P}} O_{\mathcal{P}} D_{\mathcal{P}}^{-1} O_{\mathcal{P}}^T M_{\mathcal{P},\mathcal{E}} \tilde{O}_{\mathcal{E}} \tilde{D}_{\mathcal{E}}^{-1/2}. \end{aligned}$$

8. Compute the eigenvalue decomposition (3.12) of the Schur complement matrix $M_{\mathcal{D},\mathcal{D}}$ and define the respective sub-matrices $\tilde{O}_{\mathcal{D}}^T$ and $\tilde{D}_{\mathcal{D}}$ according to (3.13) such that $\tilde{D}_{\mathcal{D}}$ is invertible.
9. Define the block-diagonal transformation

$$T_D^T := \begin{pmatrix} \mathbb{I}_{d_i^{\mathcal{P}}} & 0 \\ 0 & \tilde{D}_{\mathcal{D}}^{-1/2} \tilde{O}_{\mathcal{D}}^T \end{pmatrix}$$

from $V_i = \operatorname{span}\langle \psi_i^s, \bar{\eta}_i^t \rangle$ to $V_i = \operatorname{span}\langle \psi_i^s, \eta_i^t \rangle$.

10. Define the local preconditioner P_i^T on ω_i as $P_i^T := T_D^T \circ T_S^T \circ T_B^T$, i.e.,

$$P_i^T := \begin{pmatrix} D_{\mathcal{P}}^{-1/2} O_{\mathcal{P}}^T & 0 \\ -\tilde{D}_{\mathcal{D}}^{-1/2} \tilde{O}_{\mathcal{D}}^T M_{\mathcal{E},\mathcal{P}}^* D_{\mathcal{P}}^{-1/2} O_{\mathcal{P}}^T & \tilde{D}_{\mathcal{D}}^{-1/2} \tilde{O}_{\mathcal{D}}^T \tilde{D}_{\mathcal{E}}^{-1/2} \tilde{O}_{\mathcal{E}}^T \end{pmatrix} \tag{3.18}$$

with $M_{\mathcal{E},\mathcal{P}}^*$ given in (3.17), which maps the ill-conditioned generating system $\langle \hat{\psi}_i^s, \hat{\eta}_i^t \rangle$ to the optimally conditioned basis $\langle \psi_i^s, \eta_i^t \rangle$.

The respective global preconditioner P^T is then obtained by (3.14) and the stiffness matrix with respect to the computed stable global basis $\langle \varphi_i \vartheta_i^m \rangle = \langle \varphi_i \psi_i^s, \varphi_i \eta_i^t \rangle$ is assembled via (3.15).

The computational complexity C_P of this preconditioning scheme is given by

$$C_P := O\Big(N\big((d_i^{\mathcal{P}})^3 + (c_i^{\mathcal{E}})^3 + (d_i^{\mathcal{E}})^3\big)\Big).$$

The presented approach scales linearly in the number of patches N. With respect to the number of operations per patch ω_i the presented preconditioner is cheaper than that of [22] which scales as $O(N(d_i^{\mathcal{P}} + c_i^{\mathcal{E}})^3)$ and preserves the separability of the degrees of freedom which simplifies the error analysis.

3.1 Error Bounds

In the following we state the error bounds obtained in [22] for the sake of completeness. Due to the employed recursively defined enrichment scheme we obtain a sequence of PPUM spaces V_k^{PU} with $k = 0, \ldots, J$ that contain all polynomials up to degree $p_k = \min_i p_{i,k}$ on a particular level k and all enrichment functions η^t (up to the cut-off parameter ϵ) in the enrichment zone E on all levels k. Hence the global convergence rate of our enriched PPUM is not limited by the regularity of the solution u. To confirm this assertion we consider the splitting

$$u = u_p + \tilde{\chi}_E u_s \tag{3.19}$$

where u_p denotes the regular part of the solution u, u_s the singular part, and $\tilde{\chi}_E \in \mathcal{C}^\infty$ is a mollified characteristic function of the enrichment zone E which contains all singular points of u, i.e. of u_s. Multiplication with $1 \equiv \sum_{i=1}^N \varphi_i$ yields

$$u = \sum_{i=1}^{N} \varphi_i u_p + \sum_{i=1}^{N} \varphi_i \tilde{\chi}_E u_s.$$

If we assume that $V_i = \mathcal{P}^{p_i}$ holds for all patches ω_i with $i = 1, \ldots, M-1$ and $V_i = \mathcal{P}^{p_i} \oplus \mathcal{D}_i$ holds for all patches ω_i with $i = M, \ldots, N$ we can write an arbitrary PPUM function u^{PU} as

$$u^{\mathrm{PU}} = \sum_{i=1}^{M-1} \varphi_i \varpi_i + \sum_{i=M}^{N} \varphi_i (\varpi_i + e_i)$$

with $\varpi_i \in \mathcal{P}^{p_i}$ and $e_i \in \mathcal{D}_i$. Without loss of generality we further assume

$$\operatorname{supp}(\tilde{\chi}_E) \cap \bigcup_{i=1}^{M-1} \omega_i = \emptyset, \quad \text{i.e.} \quad \tilde{\chi}_E \sum_{i=1}^{M-1} \varphi_i \equiv 0, \tag{3.20}$$

so that we obtain the error between the analytic solution u and our PPUM approximation u^{PU} by

$$u^{\mathrm{PU}} - u = \sum_{i=1}^{M-1} \varphi_i (\varpi_i - u_p) + \sum_{i=M}^{N} \varphi_i ((\varpi_i + e_i) - (u_p + \tilde{\chi}_E u_s)). \tag{3.21}$$

By the triangle inequality we have

$$\begin{aligned}\|u - u^{\mathrm{PU}}\| \leq \| \sum_{i=1}^{M-1} \varphi_i(\varpi_i - u_p)\| \\ + \| \sum_{i=M}^{N} \varphi_i((\varpi_i + e_i) - (u_p + \tilde{\chi}_E u_s))\|.\end{aligned} \tag{3.22}$$

The first term on the right-hand side corresponds to the error of a PPUM approximation of the regular function u_p with polynomial local approximation spaces $V_i = \mathcal{P}^{p_i}$. For the ease of notation let us assume $h = \mathrm{diam}(\omega_i)$ and $p_i = 1$ for all $i = 1, \ldots, N$, then we can bound this error-term with the help of the standard PUM error analysis [2] by $O(h)$ in the H^1-norm, i.e.

$$\| \sum_{i=1}^{M-1} \varphi_i(\varpi_i - u_p)\|_{H^1} \leq O(h). \tag{3.23}$$

To obtain an upper bound for the second term of the right-hand side of (3.22)

$$J_E := \| \sum_{i=M}^{N} \varphi_i((\varpi_i + e_i) - (u_p + \tilde{\chi}_E u_s))\|$$

we consider the equality

$$u_p + \tilde{\chi}_E u_s = u_p + (\tilde{\chi}_E - 1)u_s + u_s.$$

Here the triangle inequality yields an upper bound of J_E by

$$\begin{aligned}J_E &= \| \sum_{i=M}^{N} \varphi_i((\varpi_i + e_i) - (u_p + (\tilde{\chi}_E - 1)u_s + u_s))\| \\ &\leq \| \sum_{i=M}^{N} \varphi_i(\varpi_i - (u_p + (\tilde{\chi}_E - 1)u_s))\| \\ &\quad + \| \sum_{i=M}^{N} \varphi_i(e_i - u_s)\|.\end{aligned}$$

The function $u_p + (\tilde{\chi}_E - 1)u_s$ is regular since $\tilde{\chi}_E = 1$ in the vicinity of the singular points of u_s. Hence, we can bound the first term on the right-hand side again by $O(h)$, i.e.,

$$\| \sum_{i=M}^{N} \varphi_i(\varpi_i - (u_p + (\tilde{\chi}_E - 1)u_s))\|_{H^1} \leq O(h).$$

Assuming that the enrichment functions resolve the singular part u_s of the solution u we can choose $e_i = u_s$ and so the second term vanishes and we obtain the upper bound

$$\| \sum_{i=M}^{N} \varphi_i((\varpi_i + e_i) - \tilde{\chi}_E(u_p + u_s))\|_{H^1} \leq O(h) \tag{3.24}$$

for the error in $\text{supp}(\tilde{\chi}_E) \subset E$. Together with (3.23) this yields the error bound

$$\|u - u^{\text{PU}}\|_{H_1} \leq O(h)$$

for the global error on the domain Ω.

Note however that we can obtain a better estimate for the error in the enrichment zone; i.e., our hierarchically enriched PPUM shows a kind of superconvergence within the enrichment zone. Consider the case $u_s = 0$, i.e., the approximation of a regular solution $u = u_p$ by an enriched PPUM. Then, J_E becomes

$$J_E = \| \sum_{i=M}^{N} \varphi_i((\varpi_i + e_i) - u_p))\| \tag{3.25}$$

and the standard error bound $O(h)$ ignores all degrees of freedom collected in e_i which are associated with the the enrichment functions $\eta_i^t \in \mathcal{D}_i$. Outside of the singular points of u these functions are regular and cannot be well-approximated by the polynomials $\psi_i^s \in \mathcal{P}^{p_i}$ due to our construction. Hence, the functions $\eta_i^t \in \mathcal{D}_i$ provide additional approximation power to $V_i = \mathcal{P}^{p_i} \oplus \mathcal{D}_i$ even for the approximation of smooth functions. Therefore, (3.25) can in fact be bounded by $O(h^{1+\delta})$ in the H^1-norm with $\delta > 0$ for regular solutions $u = u_p$.

For a singular solution u, i.e. $u_s \neq 0$, we can utilize this observation by considering the splitting of the enrichment part e_i on a particular patch ω_i in two local components $e_i = e_i^s + e_i^p$. On each patch ω_i with $i = M, \ldots, N$ this splitting can be chosen to balance the two error terms on the right-hand side of the inequality

$$\begin{aligned} J_E &= \| \sum_{i=M}^{N} \varphi_i((\varpi_i + e_i^p + e_i^s) - (u_p + \tilde{\chi}_E u_s))\| \\ &\leq \| \sum_{i=M}^{N} \varphi_i((\varpi_i + e_i^p) - (u_p + (\tilde{\chi}_E - 1)u_s))\| \\ &\quad + \| \sum_{i=M}^{N} \varphi_i(e_i^s - u_s)\|. \end{aligned} \tag{3.26}$$

This can yield a much smaller error bound since the regular function $u_p + (\tilde{\chi}_E - 1)u_s$ is now approximated by more degrees of freedom, i.e., by all polynomials and a number of enrichment functions η_i^t on ω_i. Yet, at the expense that $e_i^s - u_s \neq 0$. Hence, the Galerkin solution which minimizes J_E (i.e. minimizes (3.21) with respect to the energy-norm) can show a better convergence of $O(h^{1+\delta})$ with $\delta > 0$ in the enrichment zone than the global $O(h)$ behavior. The absolute value of δ is of course dependent on the specific original enrichment

functions η^t and the polynomial degree p employed since the constructed pure enrichment space is $\mathcal{D}_i = \mathcal{E}_i \setminus \mathcal{P}^{p_i}$. Note that δ is *not* related to the capability of the global enrichment functions η^t to resolve a specific singularity but rather to the approximation property of the localized enrichment functions η_i^t for regular solutions. The approximation property of the enrichment functions with respect to the singularities of the solution is required to bound the second term of the right-hand side of (3.26) and for the validity of the decomposition (3.19) with $\tilde{\chi}_E$ satisfying (3.20).

We summarize our error bounds for the enriched PPUM with local preconditioning in the following theorem.

Theorem 1. *Let $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$ be given. Let $\{\varphi_i\}$ with $\omega_i := \mathrm{supp}(\varphi_i)$ and $\mathrm{diam}(\omega_i) \asymp h$ be a partition of unity which satisfies* (2.6) *and the flat-top property according to Definition 1. Let a collection of local approximation spaces $V_i := \mathcal{P}^p \oplus \mathcal{D}_i \subset H^1(\Omega \cap \omega_i)$ with a stable basis $\langle \psi_i^s, \eta_i^t \rangle$ be given for all $i = 1, \ldots, N$ such that $\mathcal{D}_i = \{0\}$ for $i = 1, \ldots, M-1$. Assume that the spaces $\mathcal{D}_i = \mathrm{span}\langle \eta_i^t \rangle$ for $i = M, \ldots, N$ resolve all singularities of the solution $u \in H^1(\Omega)$; i.e., there holds the decomposition*

$$u = u_p + \tilde{\chi}_E u_s \quad \textit{with} \quad u_p \in H^k(\Omega)$$

and a mollified characteristic function $\tilde{\chi}_E \in \mathcal{C}^\infty$ of the enrichment zone

$$E := \bigcup_{i=M}^{N} \omega_i \quad \textit{such that} \quad \tilde{\chi}_E \sum_{i=1}^{M-1} \varphi_i \equiv 0$$

and $\tilde{E} := \mathrm{supp}(\tilde{\chi}_E) \subset E$. Then the basis $\langle \varphi_i \psi_i^s, \varphi_i \eta_i^t \rangle$ of the space

$$V^{\mathrm{PU}} := \sum_{i=1}^{N} \varphi_i V_i$$

is stable and there hold the estimates

$$\begin{aligned} \|u^{\mathrm{PU}} - u\|_{H^1(\Omega)} &\leq O(h^{\min\{p,k\}}), \\ \|u^{\mathrm{PU}} - u\|_{H^1(\Omega \setminus \tilde{E})} &\leq O(h^{\min\{p,k\}}), \\ \|u^{\mathrm{PU}} - u\|_{H^1(E)} &\leq O(h^{\min\{p,k\}+\delta}), \end{aligned}$$

for the unique best approximation $u^{\mathrm{PU}} \in V^{\mathrm{PU}}$

$$u^{\mathrm{PU}} := \sum_{i=1}^{N} \varphi_i v_i = \sum_{i=1}^{N} \varphi_i \sum_{s=1}^{d_i^{\mathcal{P}}} v_i^s \psi_i^s + \sum_{t=1}^{d_i^{\mathcal{D}}} v_i^{t+d_i^{\mathcal{P}}} \eta_i^t$$

with $\delta = \delta(p, \{\mathcal{D}_i\}) > 0$.

Remark 4. The impact of this observation is that the coefficients of the asymptotic expansion of the solution, i.e. the stress intensity factors, can be extracted from the solution with much higher accuracy and better convergence behavior in the enrichment zone than the global error bound implies. Hence, we extract the stress intensity factors *close* to the singularity in the preconditioned enriched PPUM.

4 Numerical Results

In this section we present some results of our numerical experiments using the enriched PPUM with local preconditioning as discussed above. To this end, we introduce some shorthand notation for various norms of the error $u - u^{\mathrm{PU}}$, i.e., we define

$$e_{L^\infty} := \frac{\|u - u^{\mathrm{PU}}\|_{L^\infty}}{\|u\|_{L^\infty}}, e_{L^2} := \frac{\|u - u^{\mathrm{PU}}\|_{L^2}}{\|u\|_{L^2}}, e_{H^1} := \frac{\|u - u^{\mathrm{PU}}\|_{H^1}}{\|u\|_{H^1}}. \tag{4.1}$$

For each of these error norms we compute the respective algebraic convergence rate ρ by considering the error norms of two consecutive levels $l-1$ and l

$$\rho := -\frac{\log\left(\frac{\|u-u_l^{\mathrm{PU}}\|}{\|u-u_{l-1}^{\mathrm{PU}}\|}\right)}{\log(\frac{\mathsf{dof}_l}{\mathsf{dof}_{l-1}})}, \quad \text{where } \mathsf{dof}_k := \sum_{i=1}^{N_k} \dim(V_{i,k}). \tag{4.2}$$

Hence the optimal rate ρ_{H^1} of an uniformly h-refined sequence of spaces with $p_{i,k} = p$ for all $i = 1, \ldots, N_k$ and $k = 0, \ldots, J$ for a regular solution u is $\rho_{H^1} = \frac{p}{d}$ where d denotes the spatial dimension of $\Omega \subset \mathbb{R}^d$. This corresponds to the classical $h^{\gamma_{H^1}}$ notation with $\gamma_{H^1} = \rho_{H^1} d = p$.

We consider a sequence of uniformly refined covers C_Ω^k with $\alpha = 1.3$ in (2.3) and local polynomial spaces $\mathcal{P}^{p_{i,k}} = \mathcal{P}^1$ on all levels $k = 1, \ldots, J$ in this paper. The number of patches on level k is given by $N_k = 2^{dk}$. The construction of the local preconditioners $P_{i,k}^T$ of (3.18) is based on the H^1-norm and employs a cut-off parameter $\epsilon = 10^{-12}$.

Example 1. To assess the quality of our hierarchical enrichment scheme with local preconditioning we first consider the simple model problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega = (-1,1)^2 \subset \mathbb{R}^2, \\ u &= g \quad \text{on } \partial\Omega, \end{aligned} \tag{4.3}$$

where we choose f and g such that the analytic solution u is given by

$$u(x,y) = \sqrt{r}(\sin\frac{\theta}{2} + \cos\frac{\theta}{2})(1 + \sin\theta) + (x^2 - 1) + (y^2 - 1) + 1 \tag{4.4}$$

where $r = r(x,y)$ and $\theta = \theta(x,y)$ denote polar coordinates, see Figure 2. The solution (4.4) taken from [22] is discontinuous along the line

$$C := \{(x,y) \in \overline{\Omega} \,|\, x \in [-1,0] \text{ and } y = 0\} \tag{4.5}$$

and weakly singular at the point $(0,0)$. The model problem (4.3) with the considered data f and g is essentially a scalar analogue of a linear elastic fracture mechanics problem such as (3.1). Hence, we employ the enrichment functions (3.2), (3.3), and (3.4) with respect to the crack C in our computations.

We consider the enrichment zone

$$E_{\text{tip}} := (-0.25, 0.25)^2 \subset \Omega = (-1, 1)^2 \tag{4.6}$$

on all levels $k = 0, \ldots, J$. We expect to find an $O(h)$ convergence behavior with respect to the H^1-norm globally, i.e. $\rho_{H^1} = \frac{1}{2}$. According to Theorem 1 we can obtain a better convergence behavior within the enrichment zone E_{tip}. To confirm this result we measure the errors (4.1) and respective convergence rates (4.2) not only globally on Ω but also with respect to the subdomains

$$E_1 := E_{\text{tip}} = \Big(-\frac{1}{4}, \frac{1}{4} \Big)^2, \quad E_2 := \Big(-\frac{1}{8}, \frac{1}{8} \Big)^2, \quad E_2 := \Big(-\frac{1}{16}, \frac{1}{16} \Big)^2. \tag{4.7}$$

The results obtained with our preconditioned enriched PPUM are summarized in Table 1, see also Figures 3 and 4. Besides the relative errors (4.1) and respective convergence rates (4.2) we also give the number of levels J, the number of degrees of freedom dof, and the number of patches N. From the displayed numbers we can clearly observe the anticipated optimal global convergence behavior on Ω with the rates $\rho_{L^2} = 1$ and $\rho_{H^1} = \frac{1}{2}$. Within the enrichment zone E_{tip} (i.e. for E_1, E_2, and E_3) we find convergence rates ρ_{L^2} between 1 and 1.5 for the L^2-norm. With respect to the H^1-norm we measure values between 0.75 and 1.2 for ρ_{H^1}, i.e., $\delta \geq \frac{1}{2}$ and we have close to quadratic convergence in the H^1-norm. These results are a strong indication that the improvement of the convergence behavior within the enrichment zone is not negligible but rather substantial in the H^1-norm. The local enrichment spaces (3.2) and (3.3) employed in linear elastic fracture mechanics problems provide much additional approximation power to $V_{i,k}$ for the resolution of locally regular solutions outside the singular points of u in E_{tip}.

Due to our preconditioning approach we expect that our iterative multilevel solver [14, 16] applied to the arising large sparse linear system

$$A_k \tilde{u}_k = \hat{f}_k \tag{4.8}$$

converges with a rate ρ_{ML} that is independent of the number of enriched patches $\omega_{i,k}$. To confirm this assertion we consider the convergence of the iterative update $c^{\text{PU}}_{k,\text{iter}}$ associated with the coefficient vector

$$\tilde{c}_{k,\text{iter}} := \tilde{u}_{k,\text{iter}} - \tilde{u}_{k,\text{iter}-1} \tag{4.9}$$

with respect to the L^2-norm and the energy-norm as well as the convergence of the residual vector

$$\hat{r}_{\text{iter}} := \hat{f}_k - A_k \tilde{u}_{k,\text{iter}} \tag{4.10}$$

in the l^2-norm. As initial guess for the solution of (4.8) on level k we use a random valued coefficient vector $\tilde{u}_{k,0}$ such that the associated PPUM function $u^{\text{PU}}_{k,0}$ satisfies $\|u^{\text{PU}}_{k,0}\|_{L^2} = 1$.

In Figure 5 we depict the convergence history of our iterative multilevel solver using a $V(1,1)$-cycle with block-Gauß–Seidel smoothing [14, 16]. From these plots we can clearly observe that the speed of convergence is not affected

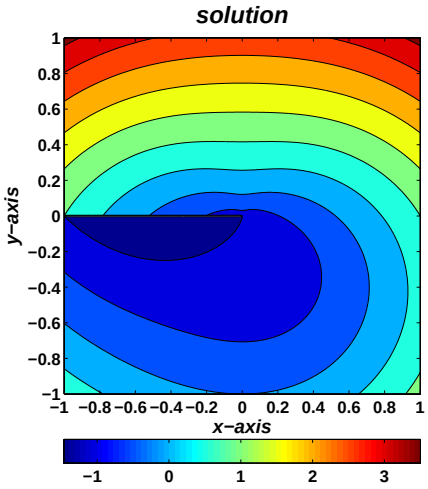

Fig. 2. Contour plot of the solution (4.4).

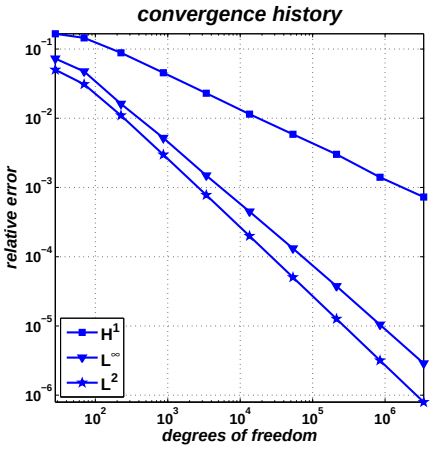

Fig. 3. Convergence history of the measured relative errors e (4.1) with respect to the complete domain Ω in the L^∞-norm, the L^2-norm, and the H^1-norm for Example 1.

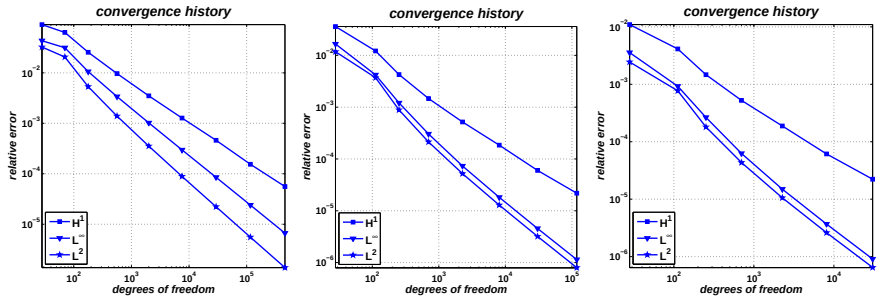

Fig. 4. Convergence history of the measured relative errors e (4.1) with respect to the subdomains E_1 (left), E_2 (center), and E_3 (right) given in (4.7) with respect to the L^∞-norm, the L^2-norm, and the H^1-norm for Example 1.

by the number of enriched patches. The number of iterations required to solve the linear system (4.8) up to machine precision is constant ≤ 20 independent of number of degrees of freedom employed in the discretization. All depicted lines are essentially parallel and have a gradient of -0.25; i.e. $\rho_{\mathrm{ML}} \approx 0.25$. This behavior clearly shows that the condition number of the preconditioned stiffness matrix is independent of the number of enriched patches.

Example 2. In our second example we consider the linear elastic fracture mechanics model problem

Table 1. Relative errors e (4.1) and convergence rates ρ (4.2) for Example 1.

J	dof	N	e_{L^∞}	ρ_{L^∞}	e_{L^2}	ρ_{L^2}	e_{H^1}	ρ_{H^1}
with respect to Ω								
1	28	4	7.262_{-2}	–	5.011_{-2}	–	1.663_{-1}	–
2	70	16	4.741_{-2}	0.47	3.096_{-2}	0.53	1.449_{-1}	0.15
3	226	64	1.614_{-2}	0.92	1.098_{-2}	0.88	8.826_{-2}	0.42
4	874	256	5.192_{-3}	0.84	2.974_{-3}	0.97	4.544_{-2}	0.49
5	3418	1024	1.488_{-3}	0.92	7.779_{-4}	0.98	2.296_{-2}	0.50
6	13498	4096	4.491_{-4}	0.87	1.990_{-4}	0.99	1.148_{-2}	0.50
7	53626	16384	1.317_{-4}	0.89	5.034_{-5}	1.00	5.864_{-3}	0.49
8	213736	65536	3.748_{-5}	0.91	1.266_{-5}	1.00	3.010_{-3}	0.48
9	853410	262144	1.042_{-5}	0.92	3.173_{-6}	1.00	1.405_{-3}	0.55
10	3410716	1048576	2.878_{-6}	0.93	7.946_{-7}	1.00	7.269_{-4}	0.48
with respect to E_1								
2	28	4	4.329_{-2}	–	3.245_{-2}	–	9.040_{-2}	–
3	70	16	3.145_{-2}	0.35	2.061_{-2}	0.50	6.313_{-2}	0.39
4	184	36	1.058_{-2}	1.13	5.325_{-3}	1.40	2.559_{-2}	0.93
5	580	100	3.385_{-3}	0.99	1.390_{-3}	1.17	9.671_{-3}	0.85
6	2044	324	1.018_{-3}	0.95	3.534_{-4}	1.09	3.495_{-3}	0.81
7	7660	1156	2.974_{-4}	0.93	8.867_{-5}	1.05	1.270_{-3}	0.77
8	29626	4356	8.505_{-5}	0.93	2.217_{-5}	1.02	4.604_{-4}	0.75
9	116532	16900	2.394_{-5}	0.93	5.538_{-6}	1.01	1.541_{-4}	0.80
10	462382	66564	6.645_{-6}	0.93	1.384_{-6}	1.01	5.586_{-5}	0.74
with respect to E_2								
3	28	4	1.659_{-2}	–	1.178_{-2}	–	3.599_{-2}	–
4	118	16	4.200_{-3}	0.95	3.707_{-3}	0.80	1.216_{-2}	0.75
5	264	36	1.209_{-3}	1.55	8.846_{-4}	1.78	4.271_{-3}	1.30
6	724	100	3.040_{-4}	1.37	2.133_{-4}	1.41	1.472_{-3}	1.06
7	2316	324	7.327_{-5}	1.22	5.209_{-5}	1.21	5.182_{-4}	0.90
8	8188	1156	1.818_{-5}	1.10	1.287_{-5}	1.11	1.849_{-4}	0.82
9	30658	4356	4.555_{-6}	1.05	3.200_{-6}	1.05	6.004_{-5}	0.85
10	118588	16900	1.141_{-6}	1.02	7.980_{-7}	1.03	2.187_{-5}	0.75
with respect to E_3								
4	28	4	3.567_{-3}	–	2.447_{-3}	–	1.096_{-2}	–
5	118	16	9.241_{-4}	0.94	7.710_{-4}	0.80	4.124_{-3}	0.68
6	264	36	2.651_{-4}	1.55	1.803_{-4}	1.80	1.471_{-3}	1.28
7	724	100	6.244_{-5}	1.43	4.337_{-5}	1.41	5.247_{-4}	1.02
8	2316	324	1.495_{-5}	1.23	1.056_{-5}	1.21	1.882_{-4}	0.88
9	8188	1156	3.685_{-6}	1.11	2.606_{-6}	1.11	6.133_{-5}	0.89
10	30652	4356	9.081_{-7}	1.06	6.476_{-7}	1.05	2.227_{-5}	0.77

$$\begin{aligned} -\mathbf{div}\,\boldsymbol{\sigma}(u) &= f \quad \text{in } \Omega \subset \mathbb{R}^d, \\ \boldsymbol{\sigma}(u)\cdot n &= g_N \text{ on } \Gamma_N, \\ u\cdot n &= g_D \text{ on } \Gamma_D = \partial\Omega \setminus \Gamma_N, \end{aligned} \tag{4.11}$$

on the domain $\Omega := (-1,1)^2$. The considered material parameters are $E = 1000$ and $\nu = 0.3$. The Dirichlet boundary is defined as

$$\Gamma_D := \{(x,y) \in \partial\Omega \,|\, y = -1\}$$

and we assume homogeneous Dirichlet data $g_D = 0$. The Neumann boundary $\Gamma_N := \partial\Omega \setminus \Gamma_D \cup C$ contains the traction-free crack C of (4.5). We apply inhomogeneous Neumann boundary conditions $g_N = 1$ on the segment $\{(x,y) \in \Gamma_N \,|\, y = 1\} \subset \Gamma_N$ only. Again we use the same enrichment zone E_{tip} given in (4.6) and the enrichment spaces (3.2), (3.3), and (3.4).

Note that we employ a reference solution $u_{\text{ref}}^{\text{PU}} \in V_{\text{ref}}^{\text{PU}}$ in this example to assess the performance of our PPUM. This reference solution was computed by

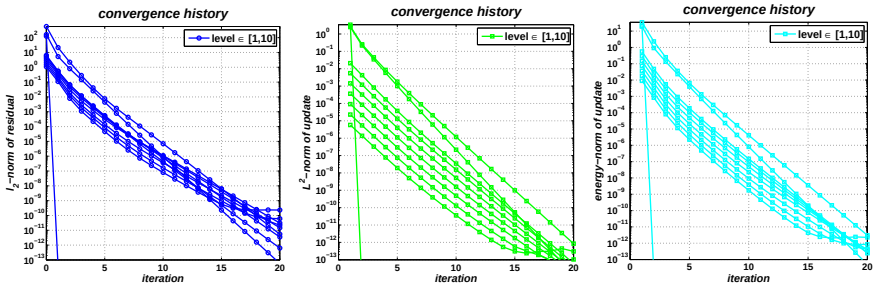

Fig. 5. Convergence history for a $V(1,1)$-cycle multilevel iteration with block-Gauß–Seidel smoother and random initial guess (left: convergence of residual vector (4.10) in the l^2-norm, center: convergence of iteration update (4.9) in the L^2-norm, right: convergence of iteration update (4.9) in the energy-norm) for Example 1.

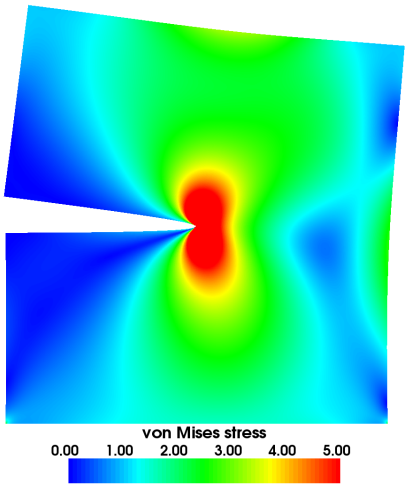

Fig. 6. Contour plot of the von Mises stress distribution depicted on the (scaled) deformed domain with respect to the reference solution $u_{\text{ref}}^{\text{PU}} \in V_{\text{ref}}^{\text{PU}}$.

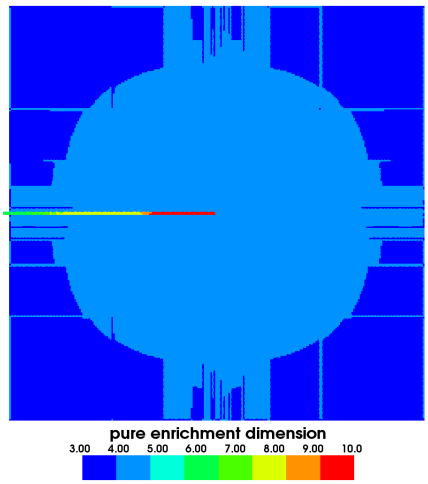

Fig. 7. Contour plot of the dimension $d_{i,J_{\text{ref}}}^{\mathcal{D}}$ of the pure enrichment spaces $\mathcal{D}_{i,J_{\text{ref}}}$ for patches $\omega_{i,J_{\text{ref}}} \subset E_{\text{tip}}$ with $V_{i,J_{\text{ref}}} = \mathcal{P}^2 \oplus \mathcal{D}_{i,J_{\text{ref}}}$ on the reference level $J_{\text{ref}} = 10$.

the enriched PPUM using a global hp-refinement of the PPUM space V_J^{PU} the finest level $J = 9$; i.e., the reference space $V_{\text{ref}}^{\text{PU}}$ employs roughly 10^6 patches, the local polynomial spaces $\mathcal{P}^2$, the enrichment spaces (3.2), (3.3), and (3.4), and has roughly 10^7 degrees of freedom $\mathsf{dof}_{\text{ref}}$. Thus, the relative errors (4.1) and respective convergence rates (4.2) given in the following refer to this reference solution $u_{\text{ref}}^{\text{PU}}$. In Figure 2 we give a contour plot of the von Mises stress of the reference solution on the deformed domain. A contour plot with respect to the enrichment zone E_{tip} of the dimension $d_{i,J_{\text{ref}}}^{\mathcal{D}} = \dim(\mathcal{D}_{i,J_{\text{ref}}})$ of the local

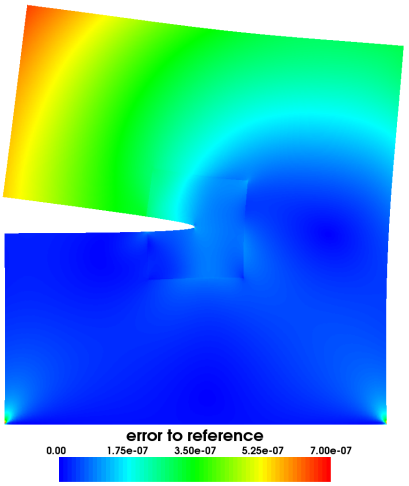

Fig. 8. Contour plot of the error on level $k = 9$ with respect to the reference solution $u^{\mathrm{PU}}_{\mathrm{ref}}$ depicted on the (scaled) deformed domain with respect to the deformation $u^{\mathrm{PU}}_k \in V^{\mathrm{PU}}_k$.

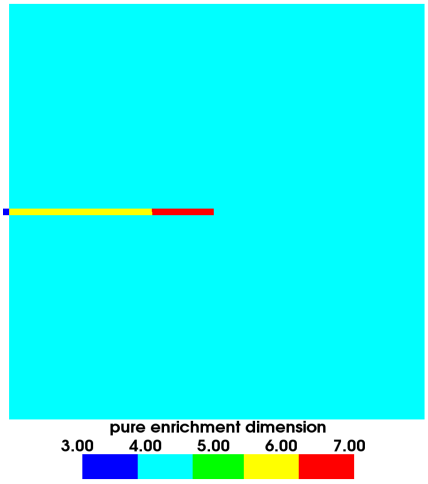

Fig. 9. Contour plot of the dimension $d^{\mathcal{D}}_{i,k}$ of the pure enrichment spaces $\mathcal{D}_{i,k}$ for patches $\omega_{i,k} \subset E_{\mathrm{tip}}$ with $V_{i,k} = \mathcal{P}^1 \oplus \mathcal{D}_{i,k}$ on level $k = 9$.

(scalar) pure enrichment spaces employed in $V^{\mathrm{PU}}_{\mathrm{ref}}$ is depicted in Figure 7. Observe that close to the singularity all original additive enrichment functions are present ($d^{\mathcal{D}}_{i,J_{\mathrm{ref}}} = \dim(W_{\mathrm{tip}}) = 4$). Further away from the singularity we find $d^{\mathcal{D}}_{i,J_{\mathrm{ref}}} = 3$; i.e. one enrichment function was automatically removed by our preconditioner. On the crack line C we employ both enrichments (3.2) and (3.3); i.e., we use (3.4), so that $c^{\mathcal{E}}_{i,J_{\mathrm{ref}}} = \dim(W_{\mathrm{tip}}) + \dim(H^C_{\pm}\mathcal{P}^2) = 4+6 = 10$. Close to the singularity all these enrichment functions are present and contribute to the approximation since we find $d^{\mathcal{D}}_{i,J_{\mathrm{ref}}} = 10$. Moving away from the singularity along on the crack the dimension $d^{\mathcal{D}}_{i,J_{\mathrm{ref}}}$ decreases and several enrichment functions were removed by our preconditioner to obtain a stable basis of $V^{\mathrm{PU}}_{\mathrm{ref}}$.

Note that there are some oscillations in the local dimension $d^{\mathcal{D}}_{i,J_{\mathrm{ref}}}$ due to the interplay of the cut-off parameter $\epsilon = 10^{-12}$ and quadrature errors in the assembly of the local operators. These oscillations however are insubstantial and have no effect on the stability, the approximation properties, or the regularity of the approximation.

We use the contour integral method (CIM) [23] to extract the stress intensity factors (SIF) $\mathsf{sif}_{\mathrm{I}}$ for mode I and $\mathsf{sif}_{\mathrm{II}}$ for mode II. For the computation of the SIFs of the reference solution we use a large number of different extraction domains $\{(x,y) \in \Omega \,|\, \max\{x,y\} < r\}$; i.e., extraction radii r, and compute the mean value of these SIFs (for $10^{-2} \leq r \leq 10^{-1}$) as our reference values

$$\mathsf{sif}^*_{\mathrm{avg,I}} = 5.091385419547622, \quad \mathsf{sif}^*_{\mathrm{avg,II}} = -0.330092517099450. \tag{4.12}$$

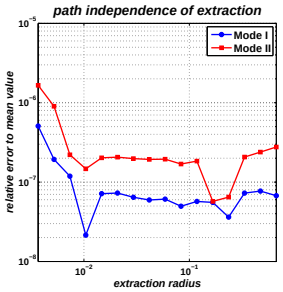

Fig. 10. Path independence of SIF extraction for reference solution.

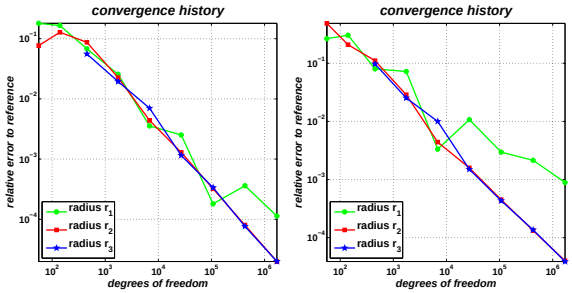

Fig. 11. Convergence history of SFIs.

Table 2. Relative errors e (4.1) and convergence rates ρ (4.2) for Example 2.

J	dof	N	e_{L^∞}	ρ_{L^∞}	e_{L^2}	ρ_{L^2}	e_{H^1}	ρ_{H^1}
with respect to Ω								
1	56	4	2.401_{-1}	–	2.028_{-1}	–	2.519_{-1}	–
2	140	16	1.945_{-1}	0.23	1.870_{-1}	0.09	2.161_{-1}	0.17
3	452	64	9.564_{-2}	0.61	9.405_{-2}	0.59	1.131_{-1}	0.55
4	1748	256	2.786_{-2}	0.91	2.761_{-2}	0.91	4.041_{-2}	0.76
5	6836	1024	7.320_{-3}	0.98	7.288_{-3}	0.98	1.608_{-2}	0.68
6	26996	4096	1.853_{-3}	1.00	1.850_{-3}	1.00	7.355_{-3}	0.57
7	107252	16384	4.635_{-4}	1.00	4.640_{-4}	1.00	3.583_{-3}	0.52
8	427472	65536	1.772_{-4}	0.70	1.160_{-4}	1.00	1.783_{-3}	0.50
9	1706820	262144	1.040_{-4}	0.39	2.901_{-5}	1.00	8.575_{-4}	0.53
with respect to E_1								
2	28	4	1.493_{-1}	–	1.448_{-1}	–	1.620_{-1}	–
3	70	16	9.055_{-2}	0.55	7.939_{-2}	0.66	1.112_{-1}	0.41
4	184	36	2.713_{-2}	1.25	2.425_{-2}	1.23	3.867_{-2}	1.09
5	580	100	7.450_{-3}	1.13	6.553_{-3}	1.14	1.249_{-2}	0.98
6	2044	324	1.954_{-3}	1.06	1.694_{-3}	1.07	4.053_{-3}	0.89
7	7660	1156	5.056_{-4}	1.02	4.331_{-4}	1.03	1.350_{-3}	0.83
8	29626	4356	1.307_{-4}	1.00	1.109_{-4}	1.01	4.610_{-4}	0.79
9	116532	16900	3.381_{-5}	0.99	2.854_{-5}	0.99	1.581_{-4}	0.78
with respect to E_2								
3	56	4	7.690_{-2}	–	7.560_{-2}	–	8.106_{-2}	–
4	236	16	2.623_{-2}	0.75	2.281_{-2}	0.83	3.331_{-2}	0.62
5	528	36	6.887_{-3}	1.66	6.297_{-3}	1.60	1.114_{-2}	1.36
6	1448	100	1.752_{-3}	1.36	1.641_{-3}	1.33	3.598_{-3}	1.12
7	4632	324	4.435_{-4}	1.18	4.224_{-4}	1.17	1.194_{-3}	0.95
8	16376	1156	1.255_{-4}	1.00	1.091_{-4}	1.07	4.064_{-4}	0.85
9	61316	4356	4.028_{-5}	0.86	2.840_{-5}	1.02	1.384_{-4}	0.82
with respect to E_3								
4	56	4	2.006_{-2}	–	2.304_{-2}	–	1.947_{-2}	–
5	236	16	7.198_{-3}	0.71	6.343_{-3}	0.90	1.296_{-2}	0.28
6	528	36	1.763_{-3}	1.75	1.675_{-3}	1.65	4.912_{-3}	1.20
7	1448	100	4.770_{-4}	1.30	4.320_{-4}	1.34	1.724_{-3}	1.04
8	4632	324	1.520_{-4}	0.98	1.117_{-4}	1.16	6.045_{-4}	0.90
9	16376	1156	4.877_{-5}	0.90	2.913_{-5}	1.06	2.089_{-4}	0.84

In Figure 10 we plot the relative error of the extracted SIFs to the mean value (for all depicted values of r). The depicted curves clearly show the path-independence of the extraction and we can infer that our reference values $\mathsf{sif}_{\mathrm{avg,I}}$ and $\mathsf{sif}_{\mathrm{avg,II}}$ are accurate with 6 digits of relative accuracy.

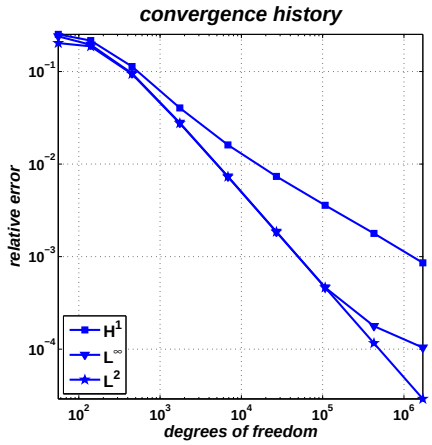

Fig. 12. Convergence history of the measured relative errors e (4.1) with respect to the complete domain Ω in the L^∞-norm, the L^2-norm, and the H^1-norm for Example 2.

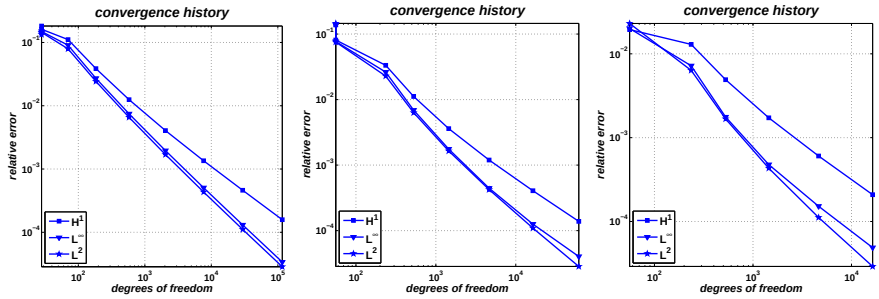

Fig. 13. Convergence history of the measured relative errors e (4.1) with respect to the subdomains E_1 (left), E_2 (center), and E_3 (right) given in (4.7) with respect to the L^∞-norm, the L^2-norm, and the H^1-norm for Example 2.

Again, we compute the relative errors and corresponding convergence rates with respect to the domain Ω and the subdomains E_1, E_2, and E_3 of (4.7). In this example we expect to find the global convergence rates to be close to $\rho_{L^2} = 1$ and $\rho_{H^1} = \frac{1}{2}$ since to solution of (4.11) has two additional but weaker singularities at the corners $(-1,-1)$ and $(1,-1)$. Within the enrichment zone we anticipate to find rates $\rho_{L^2} \geq 1$ and $\rho_{H^1} \geq \frac{1}{2}$ as in Example 1.

Table 2 summarizes the computed results. From the displayed numbers we can clearly see the anticipated convergence behavior. For the complete domain we find the rates $\rho_{L^2} = 1$ and $\rho_{H^1} = \frac{1}{2}$ and inside the enrichment zone we obtain rates between 1 and 1.5 for the L^2-norm and 0.8 and 1.2 for the H^1-norm. These results correspond very well to those of Example 1. Again, we find an almost quadratic convergence with respect to the H^1-norm within the

Table 3. Relative errors e and respective convergence rates ρ for the stress intensity factors s_{I} and s_{II} for Example 2.

J	$e_{1,\mathrm{I}}$	$\rho_{1,\mathrm{I}}$	$e_{1,\mathrm{II}}$	$\rho_{1,\mathrm{II}}$	$e_{2,\mathrm{I}}$	$\rho_{1,\mathrm{I}}$	$e_{2,\mathrm{II}}$	$\rho_{2,\mathrm{II}}$	$e_{3,\mathrm{I}}$	$\rho_{3,\mathrm{I}}$	$e_{3,\mathrm{II}}$	$\rho_{3,\mathrm{II}}$
1	1.80_{-1}	–	2.65_{-1}	–	7.68_{-2}	–	4.87_{-1}	–	–	–	–	–
2	1.65_{-1}	0.09	3.04_{-1}	−0.15	1.28_{-1}	−0.56	2.09_{-1}	0.92	–	–	–	–
3	6.79_{-2}	0.76	8.02_{-2}	1.14	8.71_{-2}	0.33	1.11_{-1}	0.54	5.57_{-2}	–	9.75_{-2}	–
4	2.56_{-2}	0.72	7.22_{-2}	0.08	2.26_{-2}	1.00	2.88_{-2}	1.00	1.94_{-2}	0.78	2.53_{-2}	1.00
5	3.56_{-3}	1.45	3.34_{-3}	2.25	4.39_{-3}	1.20	4.41_{-3}	1.38	7.01_{-3}	0.75	1.00_{-2}	0.68
6	2.52_{-3}	0.25	1.07_{-2}	−0.85	1.29_{-3}	0.89	1.59_{-3}	0.74	1.15_{-3}	1.32	1.50_{-3}	1.38
7	1.81_{-4}	1.91	2.97_{-3}	0.93	3.21_{-4}	1.01	4.51_{-4}	0.91	3.36_{-4}	0.89	4.32_{-4}	0.90
8	3.61_{-4}	−0.50	2.14_{-3}	0.24	8.01_{-5}	1.00	1.33_{-4}	0.88	7.64_{-5}	1.07	1.36_{-4}	0.83
9	1.13_{-4}	0.84	8.92_{-4}	0.63	1.99_{-5}	1.01	4.01_{-5}	0.87	1.99_{-5}	0.97	3.90_{-5}	0.90

enrichment zone E_{tip}. Observe from Figure 8 where we give a contour plot of the error on level $k = 9$ with respect to the reference solution that the error is regular (with the expected jump across the crack line). All singular components (apart from the corner singularities at $(-1,-1)$ and $(1,-1)$ which are not considered by the enrichment) of the solution are resolved. In Figure 9 we depict the dimension $d_{i,k}^{\mathcal{D}} = \dim(\mathcal{D}_{i,k})$ of the local (scalar) pure enrichment spaces employed in V_k^{PU} on level $k = 9$. Observe that our preconditioner does not remove any enrichment functions for patches $\omega_{i,k} \cap C = \emptyset$. Only along the crack line we see a decay in the dimension $d_{i,k}^{\mathcal{D}}$ moving away from the singularity at $(0,0)$.

Since the convergence of the SIFs is related to the H^1-norm we expect to find an improved convergence behavior also for the SIFs. To validate this assertion we employ three different extraction radii

$$r_1 = \frac{2}{3}, \quad r_2 = \frac{1}{6}, \quad r_3 = \frac{1}{12\sqrt{2}}, \tag{4.13}$$

where the SIFs are computed outside of E_{tip} for r_1 and inside E_{tip} for r_2 and r_3. We define the relative errors

$$e := \frac{\mathsf{sif} - \mathsf{sif}^*_{\mathrm{avg}}}{\mathsf{sif}^*_{\mathrm{avg}}}$$

with respect to the reference values (4.12) and measure the respective convergence rates analogous to (4.2). In Table 3 we give the relative errors e_1, e_2, and e_3 obtained for the three extraction radii (4.13) with respect to mode I and mode II. From the displayed numbers we can clearly observe that the SIFs converge much faster in the enrichment zone, i.e., $\rho_{1,\mathrm{I}}$ and $\rho_{1,\mathrm{II}}$ are smaller than $\rho_{r,\mathrm{I}}$ and $\rho_{r,\mathrm{II}}$ with $r = 2, 3$ respectively, see also Figure 11. The relative errors are about one order smaller inside E_{tip}. Furthermore, we find an almost perfect agreement of the SIFs computed with respect to r_2 and r_3 with an average convergence rate of about 1.

5 Concluding Remarks

We presented an efficient preconditioning technique for the hierarchically enriched PPUM. The proposed approach is applicable to any PU based enrichment scheme and is independent of the employed enrichment functions. The only prerequisite for our fully automatic algebraic construction is that the employed PU satisfies the flat-top property.

The hierarchical enrichment scheme yields approximation spaces that are capable of resolving the singular behavior of the solution completely so that the consistency of our method is not limited by the regularity of the solution. The stability of our approach is attained by the presented preconditioning technique. Hence, the PPUM with hierarchical enrichment and local preconditioning converges globally with a rate that is not limited by the smoothness of the solution. Within the enrichment zone we even obtain a kind of superconvergence due to the enrichment functions. Thus, the coefficients of the asymptotic expansion of the solution; i.e., the stress intensity factors, can be extracted from the enrichment zone with much higher accuracy and faster convergence than the global error bounds imply.

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