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Singular value decomposition versus sparse grids: Refined complexity estimates

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We compare the cost complexities of two approximation schemes for functions which live on the product domain $\Omega_1 \times \Omega_2$ of sufficiently smooth domains $\Omega_1 \subset \mathbb{R}^{n_1}$ and $\Omega_2 \subset \mathbb{R}^{n_2}$, namely the singular value / Karhunen-Lòeve decomposition and the sparse grid representation. We assume that appropriate finite element methods with associated orders r_1 and r_2 of accuracy are given on the domains Ω_1 and Ω_2, respectively. This setting reflects practical needs, since often black-box solvers are used in numerical simulation which restrict the freedom in the choice of the underlying discretization. We compare the cost complexities of the associated singular value decomposition and the associated sparse grid approximation. It turns out that, in this situation, the approximation by the sparse grid is always equal or superior to the approximation by the singular value decomposition. The results in this article improve and generalize those from Griebel & Harbrecht (2014). We now especially consider the approximation of functions from generalized isotropic and anisotropic Sobolev spaces.

Keywords: singular value decomposition; sparse grids; complexity.

1. Introduction

With this article, we intend to refine the results which have been achieved in Griebel & Harbrecht (2014), where we were concerned with the comparison of low-rank approximation methods and sparse grid methods for bivariate functions. This is a relevant setting since many problems in science and engineering lead to problems on the product $\Omega_1 \times \Omega_2$ of two domains $\Omega_1 \subset \mathbb{R}^{n_1}$ and $\Omega_2 \subset \mathbb{R}^{n_2}$. For example, radiosity models and radiative transfer (Widmer *et al.* (2008)), space-time formulations of parabolic problems (Griebel & Oeltz (2007)), phase space problems (Balescu (1997)), biscale homogenization (Cioranescu *et al.* (2008)), as well as correlation equations (Deb *et al.* (2001)) fit into this setting. We refer the reader to Griebel & Harbrecht (2014) for a more comprehensive discussion of these problems and further references.

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Moreover, the representation of functions on product domains by low-rank approximation is also the fundamental idea of reduced basis methods and model order reduction, see Hesthaven *et al.* (2016), Quarteroni *et al.* (2016), Rozza *et al.* (2008), and the references therein. Similarly, in uncertainty quantification, the spatial variable and the stochastic variable are defined on different domains. In general, after inserting the Karhunen-Lòeve decomposition of the underlying random field, one arrives at a parametric problem posed on the product of the physical domain and a high- or even infinite-dimensional parameter domain, see Ghanem & Spanos (1991) and Le Maître & Knio (2010) for example.

All the aforementioned problems are directly given on the product of two domains. Furthermore, for some of these as well as for many other problems, the domains themselves are products of lower-dimensional domains. Then, the domain of an n-dimensional problem with, for instance, n being some power of two can be split into the product of two domains of dimension $n/2$ which can recursively be further split until a terminal situation (a one-dimensional domain or a truly higher dimensional but non-tensor product domain) is reached. Related representation methods have been considered in Bebendorf (2011), Grasedyck (2010), Hackbusch (2012), Hackbusch & Kühn (2009), or Oseledets & Tyrtyshnikov (2009). Here, one should note that hierarchical tensor formats, such as the hierarchical singular value decomposition or the tensor train format, exploit a truncated singular value decomposition for each dimension separation step. An alternative approach would here be a two-dimensional sparse grid approximation in each separation step. Then, the recursive application would yield an n-dimensional sparse grid. This motivates to consider the simple case of two domains Ω_1 and Ω_2 only. Our analysis covers then also a single bisection step in the above mentioned recursion.

Our setting is as follows. We suppose to have given, fixed sequences of nested trial spaces

$$V_0^{(i)} \subset V_1^{(i)} \subset V_2^{(i)} \subset \cdots \subset L^2(\Omega_i), \quad i = 1,2, \tag{1.1}$$

on the individual subdomains, which consist of ansatz functions of approximation orders r_1 and r_2, respectively. We hence first fix the discretization and then compare the resulting algorithms. This reflects practical needs, since often black-box codes have to be used due to the implementational complexity of the underlying problems. Note at this point that our assumption is thus fundamentally different to the setting in approximation theory, where a function class is fixed and the best algorithm is sought, compare Novak & Woźniakowski (2008), Novak & Woźniakowski (2010), and Novak & Woźniakowski (2012). It also different to the *universality* point of view, where one aims at algorithms, which are almost optimal for a wide range of function classes, see Babuška (1968) and Motornyj (1974) for example.

Having the trial spaces (1.1) at hand, we can either apply the truncated singular value decomposition

$$f_M(\mathbf{x},\mathbf{y}) := \sum_{\ell=1}^{M} \sqrt{\lambda_\ell}\,\varphi_\ell(\mathbf{x})\,\psi_\ell(\mathbf{y}), \quad \mathbf{x} \in \Omega_1, \quad \mathbf{y} \in \Omega_2,$$

or the generalized sparse grid approach

$$\widehat{f}_J(\mathbf{x},\mathbf{y}) := \sum_{j_1/\sigma+j_2\sigma\leqslant J} \sum_{k_1\in\nabla_{j_1}^{(1)}} \sum_{k_2\in\nabla_{j_2}^{(2)}} \beta_{(j_1,k_1),(j_2,k_2)}\,\xi_{j_1,k_1}^{(1)}(\mathbf{x})\,\xi_{j_2,k_2}^{(2)}(\mathbf{y}), \quad \mathbf{x} \in \Omega_1, \quad \mathbf{y} \in \Omega_2$$

to represent a given function $f \in L^2(\Omega_1 \times \Omega_2)$ in an efficient way. In the first representation, $\{\varphi_\ell\}_{\ell=1}^M$ and $\{\psi_\ell\}_{\ell=1}^M$ are sets of orthonormal functions. They are a-priorily unknown, can in general not be derived analytically, and need thus to be approximated in the ansatz spaces $\{V_j^{(i)}\}$. In other words, the approximation involves in most applications both, a truncation after M terms *and* an approximate

computation of the singular values and the associated left and right singular vectors. In the second representation, $\sigma > 0$ is an appropriately chosen parameter and $\{\xi_{j,k}^{(i)}\}_{k\in\nabla_j^{(i)},j\in\mathbb{N}}$ are in general multilevel or wavelet bases associated with the trial spaces, where the index j refers to the level of resolution and the index k refers to the locality of the basis function (the precise definition will be given in Section 4). In order to decide which approximation should be implemented for treating problems on product domains, we need to know the pro's and con's of both methods.

The main improvement of our theory in comparison to Griebel & Harbrecht (2014) concerns the approximative truncated singular value decomposition. Namely, it turned out that it is not optimal to directly approximate the singular values and the eigenfunctions of the function under consideration. In this article, we therefore proceed differently: We first apply an L^2-projection on an appropriately chosen full tensor product space and then truncate the discrete singular value decomposition afterwards. This way, we are now able to directly derive sharp error estimates in the trace norm, while an approximation of the continuous eigenfunctions is no longer needed.

For our comparison, we consider the smoothness of the function f to be measured in isotropic and anisotropic Sobolev norms. We then want to compare the cost complexity to reach an approximation with a prescribed accuracy for the truncated singular value decomposition and the sparse grid approach. One result of this article is then as follows: Given a function $f \in H^p(\Omega_1 \times \Omega_2)$, we have to spend $\mathcal{O}(\varepsilon^q)$ degrees of freedom with

$$q_{svd} = \frac{\min\{n_1,n_2\}}{p} + \max\left\{\frac{\max\{n_1,n_2\}}{p}, \frac{n_1}{r_1}, \frac{n_2}{r_2}\right\}$$

for the approximation by the truncated singular value decomposition and with

$$q_{sg} = \max\left\{\frac{n_1+n_2}{p}, \frac{n_1}{r_1}, \frac{n_2}{r_2}\right\}$$

for the general sparse grid method with associated parameter $\sigma = n_1/n_2$ (a precise definition is given in Section 4), see also Griebel & Harbrecht (2013). Since it always holds

$$\frac{\min\{n_1,n_2\}}{p} + \max\left\{\frac{\max\{n_1,n_2\}}{p}, \frac{n_1}{r_1}, \frac{n_2}{r_2}\right\} \geqslant \max\left\{\frac{n_1+n_2}{p}, \frac{n_1}{r_1}, \frac{n_2}{r_2}\right\},$$

we deduce that the approximation by the sparse grid method is superior to the approximation by the singular value decomposition, at least for our setting. Moreover, we refine this result by considering in Section 5 more general isotropic and anisotropic Sobolev spaces. Also in these cases, the superiority of the sparse grid approach can be established. Recall here again that we fixed the underlying discretization via (1.1) and now compare the resulting associated algorithms.

The remainder of this article is organized as follows: In Section 2, we give a short introduction to multilevel approximation. In Section 3, we describe the singular value decomposition of a bivariate function on $\Omega_1 \times \Omega_2$ and discuss its approximation properties in detail. Section 4 gives the basics of the so-called general sparse grid approximation of a bivariate function on $\Omega_1 \times \Omega_2$ and presents its error rates and cost complexities. In Section 5, we compare the two approximations and make some final remarks.

Throughout this article, the notion "essential" in connection with the complexity estimates means "up to logarithmic terms". Moreover, to avoid the repeated use of generic but unspecified constants, we denote by $C \lesssim D$ that C is bounded by a multiple of D independently of parameters which C and D may depend on. Obviously, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \sim D$ as $C \lesssim D$ and $C \gtrsim D$.

2. Preliminaries

2.1 *Approximation on the subdomains*

Let $\Omega \subset \mathbb{R}^n$ be a sufficiently smooth, bounded domain. In general, one uses finite elements to approximate functions on $L^2(\Omega)$. In the present article, we focus on the common h-method, i.e., on finite elements of *fixed* approximation order. Then, particularly for applying multiscale techniques, one has a sequence of nested trial spaces

$$V_0 \subset V_1 \subset V_2 \subset \cdots \subset L^2(\Omega) \tag{2.1}$$

such that

$$L^2(\Omega) = \overline{\bigcup_{j\in\mathbb{N}_0} V_j},$$

which is called *multiscale analysis*. Each space V_j is defined by a single scale basis $\Phi_j = \{\phi_{j,k}\}$, i.e. $V_j = \operatorname{span}\{\phi_{j,k} : k \in \Delta_j\}$, where Δ_j denotes a suitable index set with cardinality $\#\Delta_j \sim 2^{nj}$.

We say that the trial spaces have *(approximation) order* $r \in \mathbb{N}$ if

$$r = \sup\left\{ s \in \mathbb{R} : \inf_{v_j\in V_j} \|v - v_j\|_{L^2(\Omega)} \lesssim h_j^s \|v\|_s \text{ for all } v \in H^s(\Omega) \right\}, \tag{2.2}$$

where the quantity $h_j \sim 2^{-j}$ corresponds to the mesh width associated with the subspace V_j on Ω. Note that the integer $r > 0$ refers in general to the maximal order of polynomials which are locally contained in V_j.

Equation (2.2) implies that a given function $v \in H^p(\Omega)$, $0 \leqslant p \leqslant r$, can be approximated in V_j at a rate h_j^p, i.e., the associated L^2-orthogonal projection $Q_j : L^2(\Omega) \to V_j$ satisfies

$$\|(I - Q_j)v\|_{L^2(\Omega)} \lesssim h_j^p \|v\|_{H^p(\Omega)}, \quad 0 \leqslant p \leqslant r. \tag{2.3}$$

Thus, when we approximate a function $v \in H^p(\Omega)$ with $0 \leqslant p \leqslant r$ by uniform mesh refinement we obtain the rate h_j^p according to (2.3). Since the mesh size and the number of unknowns in V_j are related by $\dim(V_j) \sim 2^{jn} \sim h_j^{-n}$, we deduce that

$$N \sim \varepsilon^{-n/p} \tag{2.4}$$

unknowns have to be spent to achieve an approximation error ε. The best possible rate $N^{-n/r}$ is achieved if $p = r$, that is if $v \in H^r(\Omega)$.

2.2 *Kolmogorov's n-width and full tensor product spaces*

For our subsequent analysis of the approximation of bivariate functions in $L^2(\Omega_1 \times \Omega_2)$, we shall fix the definitions, properties and cost complexities individually for each subdomain $\Omega_i \in \mathbb{R}^{n_i}$, $i = 1, 2$. That is, we fix two multiscale analyses

$$V_0^{(i)} \subset V_1^{(i)} \subset V_2^{(i)} \subset \cdots \subset L^2(\Omega_i), \quad i = 1, 2, \tag{2.5}$$

which are assumed to provide the approximation orders r_1 and r_2, respectively.

We start our discussion with the approximation of a given bivariate function $f \in L^2(\Omega_1 \times \Omega_2)$ in full tensor product spaces $V_{j_1}^{(1)} \otimes V_{j_2}^{(2)}$. To measure the smoothness of bivariate functions, we define the Sobolev space of dominating mixed derivatives by

$$H_{mix}^{s_1,s_2}(\Omega_1 \times \Omega_2) := H^{s_1}(\Omega_1) \otimes H^{s_2}(\Omega_2)$$

and set

$$H^{s_1,s_2}_{iso}(\Omega_1\times\Omega_2) := H^{s_1,0}_{mix}(\Omega_1\times\Omega_2)\cap H^{0,s_2}_{mix}(\Omega_1\times\Omega_2).$$

Note that, in case of $p = p_1 = p_2$, the space $H^{p,p}_{iso}(\Omega_1\times\Omega_2)$ coincides with the standard isotropic Sobolev space $H^p(\Omega_1\times\Omega_2)$, i.e., it holds

$$H^p(\Omega_1\times\Omega_2) = H^{p,p}_{iso}(\Omega_1\times\Omega_2).$$

Let $f\in H^{p_1,p_2}_{iso}(\Omega_1\times\Omega_2)$. Then, for the L^2-orthogonal projections onto $V^{(1)}_{j_1}$ and $V^{(2)}_{j_2}$, respectively, we obtain

$$\begin{aligned}\big\|(I-Q^{(1)}_{j_1}\otimes I)f\big\|_{L^2(\Omega_1\times\Omega_2)} &\lesssim 2^{-j_1\min\{p_1,r_1\}}\|f\|_{H^{\min\{p_1,r_1\},0}_{mix}(\Omega_1\times\Omega_2)},\\ \big\|(I-I\otimes Q^{(2)}_{j_2})f\big\|_{L^2(\Omega_1\times\Omega_2)} &\lesssim 2^{-j_2\min\{p_2,r_2\}}\|f\|_{H^{0,\min\{p_2,r_2\}}_{mix}(\Omega_1\times\Omega_2)}.\end{aligned} \tag{2.6}$$

Using standard tensor product arguments leads thus to

$$\begin{aligned}&\big\|(I-Q^{(1)}_{j_1}\otimes Q^{(2)}_{j_2})f\big\|_{L^2(\Omega_1\times\Omega_2)}\\ &\quad\lesssim 2^{-j_1\min\{p_1,r_1\}}\|f\|_{H^{\min\{p_1,r_1\},0}_{mix}(\Omega_1\times\Omega_2)} + 2^{-j_2\min\{p_2,r_2\}}\|f\|_{H^{0,\min\{p_2,r_2\}}_{mix}(\Omega_1\times\Omega_2)}\\ &\quad\lesssim \big(2^{-j_1\min\{p_1,r_1\}}+2^{-j_2\min\{p_2,r_2\}}\big)\|f\|_{H^{\min\{p_1,r_1\},\min\{p_2,r_2\}}_{iso}(\Omega_1\times\Omega_2)}.\end{aligned}$$

The optimum choice is to equilibrate the errors, since the approximation errors are additive while the cost are multiplicative. This means that

$$j_1\min\{p_1,r_1\}\sim j_2\min\{p_2,r_2\}, \tag{2.7}$$

which implies, in view of $\dim(V^{(1)}_{j_1})\sim 2^{j_1 n_1}$ and $\dim(V^{(2)}_{j_2})\sim 2^{j_2 n_2}$, the cost complexity

$$\mathrm{dof}_{fg}(\varepsilon) = \varepsilon^{-\frac{n_1}{\min\{p_1,r_1\}}}\varepsilon^{-\frac{n_2}{\min\{p_2,r_2\}}} \tag{2.8}$$

to achieve a desired approximation error ε. If $p_1\leqslant r_1$ and $p_2\leqslant r_2$, this is known to be Kolmogorov's n-width for Sobolev balls in the space $H^{p_1,p_2}_{iso}(\Omega_1\times\Omega_2)$, see Kolmogorov (1936). Hence, the cost complexity (2.8) is sharp in this case, which means, there is no better representation possible. Nonetheless, if $p_1 > r_1$ or $p_2 > r_2$, then (2.8) is not sharp anymore and we can approximate better in $H^{p_1,p_2}_{iso}(\Omega_1\times\Omega_2)$ than by just using the full tensor product space.

The methods we discuss in this article are the approximative truncated singular value decomposition in Section 3 and the sparse grid in Section 4. The question we address is as follows: Given a function $H^{p_1,p_2}_{iso}(\Omega_1\times\Omega_2)$, where $p_1,p_2>0$ are arbitrary and where trial spaces with approximation orders r_1 and r_2, respectively, are used in both approaches, which algorithm provides the cheaper approximation?

3. Singular value decomposition

3.1 *Definition and mapping properties*

We intend to numerically represent functions $f\in L^2(\Omega_1\times\Omega_2)$ on tensor product domains $\Omega_1\times\Omega_2$ in an efficient way. One way to solve this approximation problem is to use an ansatz by means of tensor products which separates the variables $\mathbf{x}\in\Omega_1$ and $\mathbf{y}\in\Omega_2$. We first consider the approximation

$$f(\mathbf{x},\mathbf{y})\approx f_M(\mathbf{x},\mathbf{y}) = \sum_{\ell=1}^{M}\alpha_\ell\varphi_\ell(\mathbf{x})\psi_\ell(\mathbf{y}) \tag{3.1}$$

with certain coefficients $\alpha_\ell \in \mathbb{R}$ and normalized functions $\varphi_\ell \in L^2(\Omega_1)$ and $\psi_\ell \in L^2(\Omega_2)$. Such an approximation is called *low-rank approximation*.

It is well known (see e.g. Ghanem & Spanos (1991) or Lòeve (1978)) that, with respect to the number M of terms, the best possible representation of a function $f \in L^2(\Omega_1 \times \Omega_2)$ in the L^2-sense is given by the Karhunen-Lòeve / singular value decomposition. Then, $\alpha_\ell = \sqrt{\lambda_\ell}$ are given by the eigenvalues of the below defined integral operator (3.2) with kernel (3.3). As shown in Subsection 3.2, the truncation error (in terms of M) of the series (3.1) is related to the smoothness of the function f to be approximated. As a byproduct of this estimate, we can infer the decay of the eigenvalues in Subsection 3.3. In Subsection 3.4, we finally consider the numerical treatment of (3.1). Besides determining the coefficients $\{\alpha_\ell\}_{\ell\in\mathbb{N}}$, a numerical scheme needs to approximate the functions $\{\varphi_\ell\}_{\ell\in\mathbb{N}}$ and $\{\psi_\ell\}_{\ell\in\mathbb{N}}$ in appropriate trial spaces $V_{j_1}^{(1)}$ and $V_{j_2}^{(2)}$, respectively, up to an accuracy corresponding to that of (3.1). Recall that the trial spaces which we consider are elements of the multiscale analyses (2.5) which have the approximation orders r_1 and r_2, respectively.

To derive the singular value decomposition, we shall consider the integral operator

$$\mathscr{S} : L^2(\Omega_1) \to L^2(\Omega_2), \quad u \mapsto (\mathscr{S}u)(\mathbf{y}) := \int_{\Omega_1} f(\mathbf{x},\mathbf{y})u(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

Its adjoint is

$$\mathscr{S}^\star : L^2(\Omega_2) \to L^2(\Omega_1), \quad u \mapsto (\mathscr{S}^\star u)(\mathbf{x}) := \int_{\Omega_2} f(\mathbf{x},\mathbf{y})u(\mathbf{y})\,\mathrm{d}\mathbf{y}.$$

To obtain the low-rank representation (3.1), we need to compute the eigenvalues of the integral operator

$$\mathscr{K} = \mathscr{S}^\star\mathscr{S} : L^2(\Omega_1) \to L^2(\Omega_1), \quad u \mapsto (\mathscr{K}u)(\mathbf{x}) := \int_{\Omega_1} k(\mathbf{x},\mathbf{x}')u(\mathbf{x}')\,\mathrm{d}\mathbf{x}' \tag{3.2}$$

whose kernel function is given by

$$k(\mathbf{x},\mathbf{x}') = \int_{\Omega_2} f(\mathbf{x},\mathbf{y})f(\mathbf{x}',\mathbf{y})\,\mathrm{d}\mathbf{y} \in L^2(\Omega_1 \times \Omega_1). \tag{3.3}$$

This is a Hilbert-Schmidt kernel. Thus, the associated integral operator $\mathscr{K}$ is compact. Moreover, since $\mathscr{K}$ is self-adjoint, there exists a decomposition into eigenpairs $(\lambda_\ell, \varphi_\ell)$, i.e.,

$$\mathscr{K}\varphi_\ell = \lambda_\ell\varphi_\ell \quad \text{for all } \ell \in \mathbb{N},$$

with non-negative eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m \to 0$ and eigenfunctions $\{\varphi_\ell\}_{\ell\in\mathbb{N}}$, which constitute an orthonormal basis in $L^2(\Omega_1)$.

We now define for all $\ell \in \mathbb{N}$ with $\lambda_\ell > 0$ the function $\psi_\ell \in L^2(\Omega_2)$ by

$$\psi_\ell(\mathbf{y}) = \frac{1}{\sqrt{\lambda_\ell}}(\mathscr{S}\varphi_\ell)(\mathbf{y}) = \frac{1}{\sqrt{\lambda_\ell}}\int_{\Omega_1} f(\mathbf{x},\mathbf{y})\varphi_\ell(\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{3.4}$$

This constitutes a second sequence of orthonormal functions since

$$\begin{aligned}(\psi_k,\psi_\ell)_{L^2(\Omega_2)} &= \frac{1}{\sqrt{\lambda_k\lambda_\ell}}(\mathscr{S}\varphi_k,\mathscr{S}\varphi_\ell)_{L^2(\Omega_2)} = \frac{1}{\sqrt{\lambda_k\lambda_\ell}}(\mathscr{K}\varphi_k,\varphi_\ell)_{L^2(\Omega_1)} \\ &= \frac{\lambda_k}{\sqrt{\lambda_k\lambda_\ell}}(\varphi_k,\varphi_\ell)_{L^2(\Omega_1)} = \delta_{k,\ell}.\end{aligned}$$

If $\lambda_\ell = 0$ for some $\ell \in \mathbb{N}$, we can extend this collection of functions properly to obtain an orthonormal basis $\{\psi_\ell\}_{\ell\in\mathbb{N}}$ of $L^2(\Omega_2)$. Due to

$$\sqrt{\lambda_\ell}\varphi_\ell(\mathbf{x}) = \frac{1}{\sqrt{\lambda_\ell}}(\mathscr{S}^\star\mathscr{S}\varphi_\ell)(\mathbf{x}) = (\mathscr{S}^\star\psi_\ell)(\mathbf{x}) = \int_{\Omega_2} f(\mathbf{x},\mathbf{z})\psi_\ell(\mathbf{z})\,\mathrm{d}\mathbf{z} \tag{3.5}$$

for all $\mathbf{x} \in \Omega_1$ and $\ell \in \mathbb{N}$, we finally obtain the representation

$$f(\mathbf{x},\mathbf{y}) = \sum_{\ell=1}^{\infty} \sqrt{\lambda_\ell}\varphi_\ell(\mathbf{x})\psi_\ell(\mathbf{y}). \tag{3.6}$$

With (3.4) and (3.5), this equation is easily verified by testing with the orthonormal basis $\{\varphi_k \otimes \psi_\ell\}_{k,\ell\in\mathbb{N}}$ of $L^2(\Omega_1 \times \Omega_2)$.

REMARK 3.1 The adjoint kernel $\widetilde{k}(\cdot,\cdot)$ is just obtained by interchanging Ω_1 and Ω_2, i.e.,

$$\widetilde{k}(\mathbf{y},\mathbf{y}') = \int_{\Omega_1} f(\mathbf{x},\mathbf{y})f(\mathbf{x},\mathbf{y}')\,\mathrm{d}\mathbf{x} \in L^2(\Omega_2 \times \Omega_2).$$

Then, one has the integral operator

$$\widetilde{\mathscr{K}} = \mathscr{S}\mathscr{S}^\star : L^2(\Omega_2) \to L^2(\Omega_2), \quad u \mapsto (\widetilde{\mathscr{K}}u)(\mathbf{y}) := \int_{\Omega_2} \widetilde{k}(\mathbf{y},\mathbf{y}')u(\mathbf{y}')\,\mathrm{d}\mathbf{y}'.$$

Again there exists a decomposition into eigenpairs

$$\widetilde{\mathscr{K}}\widetilde{\varphi}_\ell = \widetilde{\lambda}_\ell\widetilde{\varphi}_\ell, \quad \ell \in \mathbb{N},$$

with non-negative eigenvalues $\widetilde{\lambda}_1 \geqslant \widetilde{\lambda}_2 \geqslant \cdots \geqslant \widetilde{\lambda}_m \to 0$ and eigenfunctions $\widetilde{\varphi}_\ell \in L^2(\Omega_2)$. We also obtain a second sequence of orthonormal functions $\widetilde{\psi}_\ell \in L^2(\Omega_1)$ analogously to (3.4). The functions $\{\widetilde{\varphi}_\ell\}_{\ell\in\mathbb{N}}$ and $\{\widetilde{\psi}_\ell\}_{\ell\in\mathbb{N}}$ will be the same as before but now their roles are exchanged. Moreover, the eigenvalues λ_ℓ and $\widetilde{\lambda}_\ell$ of $\mathscr{K}$ and $\widetilde{\mathscr{K}}$ coincide.

3.2 *Truncation error*

We shall now give improved estimates on the the decay rate of the eigenvalues of the integral operator $\mathscr{K} = \mathscr{S}^\star\mathscr{S}$ with kernel (3.3). To this end, assume that $f \in H^{p,0}_{mix}(\Omega_1 \times \Omega_2)$. We introduce *new*[1] finite element spaces $U_M \subset L^2(\Omega_1)$, which consist of M discontinuous, piecewise polynomial functions of total degree $\lceil p \rceil$ on a quasi-uniform triangulation of Ω_1 with mesh width $h_M \sim M^{-1/n_1}$. Then, due to the Bramble-Hilbert lemma (see e.g., Braess (2001) or Brenner & Scott (2008)), given a function $w \in H^p(\Omega_1)$, the L^2-orthogonal projection $P_M : L^2(\Omega_1) \to U_M$ satisfies

$$\|(I - P_M)w\|_{L^2(\Omega_1)} \leqslant c_p M^{-p/n_1}\|w\|_{H^p(\Omega_1)}, \tag{3.7}$$

uniformly in M. For the approximation of $f(\mathbf{x},\mathbf{y})$ in the first variable, i.e.

$$f_M(\mathbf{x},\mathbf{y}) := \big((P_M \otimes I)f\big)(\mathbf{x},\mathbf{y}),$$

we obtain the following approximation result in U_M, see also Harbrecht *et al.* (2015).

[1]The present argument relies on an approximation argument. The new finite element spaces $\{U_M\}$ are introduced to obtain the optimal convergence rate with N degrees of freedom.

THEOREM 3.1 Let $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant 0$ be the eigenvalues of the operator $\mathscr{K} = \mathscr{S}\mathscr{S}^\star$ and $\lambda_1^M \geqslant \lambda_2^M \geqslant \ldots \geqslant \lambda_M^M \geqslant 0$ those of $\mathscr{K}_M := P_M \mathscr{K} P_M$. Then, it holds

$$\|f - f_M\|^2_{L^2(\Omega_1\times\Omega_2)} = \operatorname{trace}\mathscr{K} - \operatorname{trace}\mathscr{K}_M$$

and therefore

$$\|f - f_M\|^2_{L^2(\Omega_1\times\Omega_2)} = \sum_{\ell=1}^{M}(\lambda_\ell - \lambda_\ell^M) + \sum_{\ell=M+1}^{\infty}\lambda_\ell. \tag{3.8}$$

Proof. Let $\{\theta_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of $L^2(\Omega_1)$ such that either $\theta_k \in \operatorname{img} P_M$ or $\theta_k \in \operatorname{img}(I-P_M)$ holds. This implies $\big(\mathscr{S}(I-P_M)\theta_k, \mathscr{S}P_M\theta_k\big)_{L^2(\Omega_2)} = 0$ for all $k \in \mathbb{N}$. We thus arrive at

$$\|f - f_M\|^2_{L^2(\Omega_1\times\Omega_2)} = \int_{\Omega_2}\sum_{\ell=1}^{\infty}\big((f-f_M)(\cdot,\mathbf{y}),\theta_\ell\big)^2_{L^2(\Omega_1)}\,\mathrm{d}\mathbf{y}.$$

Due to the fact that $I - P_M$ is an L^2-orthogonal projection, we have

$$\big((f-f_M)(\cdot,\mathbf{y}),\theta_\ell\big)_{L^2(\Omega_1)} = \big(f(\cdot,\mathbf{y}),(I-P_M)\theta_\ell\big)_{L^2(\Omega_1)}.$$

Hence, it holds

$$\|f - f_M\|^2_{L^2(\Omega_1\times\Omega_2)} = \sum_{\ell=1}^{\infty}\int_{\Omega_2}\big(f(\cdot,\mathbf{y}),(I-P_M)\theta_\ell\big)^2_{L^2(\Omega_1)}\,\mathrm{d}\mathbf{y}.$$

Inserting next the definition of $\mathscr{S}$, we obtain

$$\|f - f_M\|^2_{L^2(\Omega_1\times\Omega_2)} = \sum_{\ell=1}^{\infty}\|\mathscr{S}(I-P_M)\theta_\ell\|^2_{L^2(\Omega_2)} = \sum_{\ell=1}^{\infty}\|\mathscr{S}\theta_\ell\|^2_{L^2(\Omega_2)} - \sum_{\ell=1}^{\infty}\|\mathscr{S}P_M\theta_\ell\|^2_{L^2(\Omega_2)}.$$

Finally, by using

$$\operatorname{trace}\mathscr{K} = \sum_{\ell=1}^{\infty}(\mathscr{K}\theta_\ell,\theta_\ell)_{L^2(\Omega_1)} = \sum_{\ell=1}^{\infty}\|\mathscr{S}\theta_\ell\|^2_{L^2(\Omega_1)}$$

and

$$\operatorname{trace}\mathscr{K}_M = \sum_{\ell=1}^{\infty}(\mathscr{K}_M\theta_\ell,\theta_\ell)_{L^2(\Omega_1)} = \sum_{\ell=1}^{\infty}\|\mathscr{S}P_M\theta_\ell\|^2_{L^2(\Omega_1)},$$

we conclude the assertion. □

By combining this theorem with the approximation estimate (3.7), we can obviously bound the trace error by

$$0 \leqslant \operatorname{trace}\mathscr{K} - \operatorname{trace}\mathscr{K}_M \lesssim M^{-\frac{2p}{n_1}}\|f\|^2_{H^{p,0}_{mix}(\Omega_1\times\Omega_2)}. \tag{3.9}$$

This estimate now allows to prove the following result on the truncation of the singular value decomposition after M terms.

THEOREM 3.2 Let $f \in H^{p,0}_{mix}(\Omega_1\times\Omega_2)$. Then, it holds

$$\left\|f - \sum_{\ell=0}^{M}\sqrt{\lambda_\ell}(\varphi_\ell\otimes\psi_\ell)\right\|_{L^2(\Omega_1\times\Omega_2)} \lesssim M^{-\frac{p}{n_1}}\|f\|_{H^{p,0}_{mix}(\Omega_1\times\Omega_2)}. \tag{3.10}$$

Proof. Due to the orthonormality of the sequences $\{\varphi_\ell\}$ and $\{\psi_\ell\}$ in $L^2(\Omega_1)$ and $L^2(\Omega_2)$, respectively, the error when truncating the singular value decomposition after M terms is given by

$$\left\| f - \sum_{\ell=0}^{M} \sqrt{\lambda_\ell}(\varphi_\ell \otimes \psi_\ell) \right\|^2_{L^2(\Omega_1\times\Omega_2)} = \left\| \sum_{\ell=M+1}^{\infty} \sqrt{\lambda_\ell}(\varphi_\ell \otimes \psi_\ell) \right\|^2_{L^2(\Omega_1\times\Omega_2)} = \sum_{\ell=M+1}^{\infty} \lambda_\ell.$$

In view of Theorem 3.1 and (3.9), since $\lambda_\ell \geqslant \lambda_\ell^M$ for all $\ell \in \{1,\dots,M\}$ (see e.g. Babuška & Osborn (1991)), we immediately arrive at the following estimate:

$$\sum_{\ell=M+1}^{\infty} \lambda_\ell \lesssim M^{-\frac{2p}{n_1}} \|f\|^2_{H^{p,0}_{mix}(\Omega_1\times\Omega_2)}. \tag{3.11}$$

□

According to Theorem 3.2, we only need the smoothness of f in the first coordinate to derive estimate (3.10). Since the eigenvalues of integral operator $\mathscr{K}$ and its adjoint $\widetilde{\mathscr{K}}$ are the same, we can also exploit any smoothness of f in the second coordinate, if provided, by interchanging the roles of Ω_1 and Ω_2 in the above proof. We thus obtain the following corollary.

COROLLARY 3.1 For $f \in H^{p_1,p_2}_{iso}(\Omega_1 \times \Omega_2)$, it holds

$$\left\| f - \sum_{\ell=0}^{M} \sqrt{\lambda_\ell}(\varphi_\ell \otimes \psi_\ell) \right\|_{L^2(\Omega_1\times\Omega_2)} \lesssim M^{-\max\{\frac{p_1}{n_1},\frac{p_2}{n_2}\}} \|f\|_{H^{p_1,p_2}_{iso}(\Omega_1\times\Omega_2)}. \tag{3.12}$$

Altogether, in order to ensure the bound

$$\left\| f - \sum_{\ell=0}^{M} \sqrt{\lambda_\ell}(\varphi_\ell \otimes \psi_\ell) \right\|_{L^2(\Omega_1\times\Omega_2)} \lesssim \varepsilon \tag{3.13}$$

on the truncation error of the singular value decomposition, we need, as a consequence of Theorem 3.2 and Corollary 3.1, to choose the expansion degree M in accordance with

$$M \sim \varepsilon^{-\min\{\frac{n_1}{p_1},\frac{n_2}{p_2}\}} \tag{3.14}$$

if $f \in H^{p_1,p_2}_{iso}(\Omega_1 \times \Omega_2)$. Note that in the situation $p = p_1 = p_2$, which means $f \in H^p(\Omega_1 \times \Omega_2)$, the truncation rank is simply given by $M \sim \varepsilon^{-\frac{\min\{n_1,n_2\}}{p}}$.

3.3 *Decay of the eigenvalues*

Having Corollary 3.1 at hand, we can give a bound on the decay of the singular values. Note that this estimate improves the bound

$$\lambda_\ell \lesssim \ell^{-2\min\{\frac{p_1}{n_1},\frac{p_2}{n_2}\}} \quad \text{as } \ell \to \infty$$

from Griebel & Harbrecht (2014) by an additive factor 1 in the exponent. In particular, the new bound below is now sharp (see, for instance, the specific examples in Griebel & Harbrecht (2014)). Moreover, it coincides with the bound which was derived in Dölz *et al.* (2017) and Griebel & Li (2017) by different techniques.

PROPOSITION 3.3 Consider $f \in H^{p_1,p_2}_{iso}(\Omega_1 \times \Omega_2)$ with associated kernel k from (3.3) and associated integral operator $\mathscr{K}$ from (3.2). Then, the eigenvalues $\{\lambda_\ell\}_{\ell\in\mathbb{N}}$ of $\mathscr{K}$ decay like

$$\lambda_\ell \lesssim \ell^{-2\min\{\frac{p_1}{n_1},\frac{p_2}{n_2}\}-1} \quad \text{as } \ell \to \infty.$$

Proof. Since the sequence $\{\lambda_\ell\}$ decreases monotonically, it holds on the one hand

$$\sum_{m=1}^{2^{k+1}-1} \lambda_m = \lambda_1 + (\lambda_2+\lambda_3) + \cdots + (\lambda_{2^k} + \lambda_{2^k+1} + \cdots + \lambda_{2^{k+1}-1}) \leqslant \sum_{n=0}^{k} 2^n \lambda_{2^n},$$

and on the other hand

$$\sum_{m=1}^{2^k} \lambda_m = \lambda_1 + \lambda_2 + (\lambda_3+\lambda_4) + \cdots + (\lambda_{2^{k-1}+1} + \lambda_{2^{k-1}+2} + \cdots + \lambda_{2^k}) \geqslant \frac{1}{2} \sum_{n=0}^{k} 2^n \lambda_{2^n}.$$

This is the well-known Cauchy condensation test, which implies

$$\sum_{m=2^k}^{2^{k+1}-1} \lambda_m \sim \sum_{n=0}^{k} 2^n \lambda_{2^n}.$$

For arbitrary $k \in \mathbb{N}$, in view of (3.11), we conclude with $\beta := 2\min\{\frac{p_1}{n_1}, \frac{p_2}{n_2}\}$ that

$$\sum_{n=0}^{k} 2^n \lambda_{2^n} \sim \sum_{m=2^k}^{2^{k+1}} \lambda_m \lesssim \sum_{m=2^k}^{\infty} \lambda_m \lesssim 2^{-\beta k}.$$

This, however, leads to

$$\sum_{n=0}^{k} 2^{(\beta+1)n} \lambda_{2^n} \lesssim 2^{\beta k} \sum_{m=2^k}^{2^{k+1}} \lambda_m \lesssim 2^{\beta k} \sum_{m=2^k}^{\infty} \lambda_m \lesssim 1$$

uniformly in $k \in \mathbb{N}$. Therefore, $2^{(\beta+1)n}\lambda_{2^n}$ tends to zero, which immediately implies that $\ell^{\beta+1}\lambda_\ell$ also tends to zero since

$$2^{(\beta+1)n}\lambda_{2^n} \sim \ell^{\beta+1}\lambda_\ell \sim 2^{(\beta+1)(n+1)}\lambda_{2^{n+1}} \text{ for all } 2^n \leqslant \ell < 2^{n+1}.$$

□

3.4 *Numerical approximation*

In Corollary 3.1, we used an *exact* description of the eigenfunctions. However, this does not hold in practice. Instead, the eigenvalues $\{\lambda_\ell\}_{\ell=1}^M$ and eigenfunctions $\{\varphi_\ell\}_{\ell=1}^M$ and $\{\psi_\ell\}_{\ell=1}^M$ need to be approximately computed in the finite element spaces which have been introduced in Subsection 2.1.

For functions $f \in H^{p_1,p_2}_{iso}(\Omega_1 \times \Omega_2)$ with $p_1 \leqslant r_1$ and $p_2 \leqslant r_2$, we already know that the full tensor product space yields the best possible approximation. Indeed, the singular values then decay not fast enough in order to benefit from additional compression. Hence, we shall assume $p_1 > r_1$ or $p_2 > r_2$ in the subsequent discussion.

In the following, we first consider the projection of f onto a suitable full tensor product ansatz space and perform afterwards a projection onto M dominant eigenpairs of the projected function.

When choosing the levels of refinement j_1 and j_2 for the projections $Q_{j_1}^{(1)}$ and $Q_{j_2}^{(2)}$ in (2.6), we obtain $N_\varphi = \dim(V_{j_1}^{(1)}) \sim 2^{j_1 n_1}$ and $N_\psi = \dim(V_{j_2}^{(2)}) \sim 2^{j_2 n_2}$ degrees of freedoms, respectively. Then, for $\mathscr{K}_{N_\varphi} := Q_j^{(1)} \mathscr{K} Q_j^{(1)}$ with eigenvalues $\lambda_{1,N_\varphi} \geqslant \lambda_{2,N_\varphi} \geqslant \cdots \geqslant \lambda_{N_\varphi,N_\varphi} \geqslant 0$, it holds in complete analogy to Theorem 3.1

$$\begin{aligned} \left\|(I - Q_{j_1}^{(1)} \otimes I) f\right\|_{L^2(\Omega_1\times\Omega_2)}^2 &= \operatorname{trace} \mathscr{K} - \operatorname{trace} \mathscr{K}_{N_\varphi} \\ &= \sum_{\ell=1}^{N_\varphi} (\lambda_\ell - \lambda_{\ell,N_\varphi}) + \sum_{\ell=N_\varphi+1}^{\infty} \lambda_\ell \\ &\lesssim 2^{-2 j_1 \min\{p_1,r_1\}} \|f\|_{H_{mix}^{\min\{p_1,r_1\},0}(\Omega_1\times\Omega_2)}^2. \end{aligned} \tag{3.15}$$

We emphasize again that there holds $\lambda_\ell \geqslant \lambda_{\ell,N_\varphi}$ for all $\ell \in \{1,2,\ldots,N_\varphi\}$.

For $M \leqslant N_\varphi$, let $P_M^{(1)}$ denote the projection onto the M dominant eigenpairs $(\lambda_{1,N_\varphi}, \varphi_{1,N_\varphi}), \ldots, (\lambda_{M,N_\varphi}, \varphi_{M,N_\varphi})$ of $\mathscr{K}_{N_\varphi}$, i.e.,

$$P_M^{(1)} : L^2(\Omega_1) \to L^2(\Omega_1), \quad g \mapsto P_M^{(1)} g = \sum_{k=1}^{M} (g, \varphi_{k,N_\varphi})_{L^2(\Omega_1)} \varphi_{k,N_\varphi}.$$

Then, we have

$$\begin{aligned} &\left\|(I - P_M^{(1)} \otimes I) f\right\|_{L^2(\Omega_1\times\Omega_2)}^2 \\ &\quad = \operatorname{trace} \mathscr{K} - \operatorname{trace} P_M^{(1)} \mathscr{K}_{N_\varphi} P_M^{(1)} \\ &\quad = \sum_{\ell=1}^{M} (\lambda_\ell - \lambda_{\ell,N_\varphi}) + \sum_{\ell=M+1}^{\infty} \lambda_\ell \\ &\quad \lesssim 2^{-2 j_1 \min\{p_1,r_1\}} \|f\|_{H_{mix}^{\min\{p_1,r_1\},0}(\Omega_1\times\Omega_2)}^2 + M^{-2\max\{\frac{p_1}{n_1},\frac{p_2}{n_2}\}} \|f\|_{H_{iso}^{p_1,p_2}(\Omega_1\times\Omega_2)}^2, \end{aligned} \tag{3.16}$$

where we used (3.11) and

$$0 \leqslant \sum_{\ell=1}^{M} (\lambda_\ell - \lambda_{\ell,N_\varphi}) \leqslant \sum_{\ell=1}^{N_\varphi} (\lambda_\ell - \lambda_{\ell,N_\varphi}) \leqslant 2^{-2 j_1 \min\{r_1,p_1\}} \|f\|_{H_{mix}^{\min\{r_1,p_1\},0}(\Omega_1\times\Omega_2)}^2$$

in accordance with (3.15).

We proceed in complete analogy in the second variable. For the projection

$$P_M^{(2)} : L^2(\Omega_2) \to L^2(\Omega_2), \quad g \mapsto P_M^{(2)} g = \sum_{k=1}^{M} (g, \psi_{k,N_\psi})_{L^2(\Omega_1)} \psi_{k,N_\psi}$$

onto the $M \leqslant N_\psi$ dominant eigenpairs $(\widetilde{\lambda}_{1,N_\psi}, \psi_{1,N_\psi}), \ldots, (\widetilde{\lambda}_{M,N_\psi}, \psi_{M,N_\psi})$ of $\widetilde{\mathscr{K}_{N_\psi}} := Q_{j_2}^{(2)} \widetilde{\mathscr{K}} Q_{j_2}^{(2)}$, we obtain

$$\begin{aligned} &\left\|(I - I \otimes P_M^{(2)}) f\right\|_{L^2(\Omega_1\times\Omega_2)}^2 \\ &\quad \lesssim 2^{-2 j_2 \min\{p_2,r_2\}} \|f\|_{H_{mix}^{0,\min\{p_2,r_2\}}(\Omega_1\times\Omega_2)}^2 + M^{-2\min\{\frac{p_1}{n_1},\frac{p_2}{n_2}\}} \|f\|_{H_{iso}^{p_1,p_2}(\Omega_1\times\Omega_2)}^2. \end{aligned} \tag{3.17}$$

We now choose the levels j_1 and j_2 such that (2.7) holds. Then, by combining (3.16) and (3.17), we arrive for any $M \leqslant \max\{N_\varphi, N_\psi\}$ at

$$\begin{aligned}&\left\|(I - P_M^{(1)} \otimes P_M^{(2)})f\right\|^2_{L^2(\Omega_1\times\Omega_2)}\\ &\quad\lesssim 2^{-2j_1\min\{p_1,r_1\}}\|f\|^2_{H^{\min\{p_1,r_1\},0}_{mix}(\Omega_1\times\Omega_2)} + M^{-2\min\{\frac{p_1}{n_1},\frac{p_2}{n_2}\}}\|f\|^2_{H^{p_1,p_2}_{iso}(\Omega_1\times\Omega_2)}.\end{aligned}$$

Note that the expression $(P_M^{(1)} \otimes P_M^{(2)})f$ corresponds to the desired series expansion

$$(P_M^{(1)} \otimes P_M^{(2)})f = \sum_{\ell=1}^{M} \alpha_\ell(\varphi_{\ell,N_\varphi} \otimes \psi_{\ell,N_\psi}),$$

where the coefficients $\{\alpha_\ell\}$ are given by

$$\alpha_\ell := (f, \varphi_{\ell,N_\varphi} \otimes \psi_{\ell,N_\psi})_{L^2(\Omega_1\times\Omega_2)} \quad \text{for all } \ell \in \{1,2,\ldots,M\}.$$

The approximation error is already fixed by the projection onto the full tensor product space. Hence, it remains to equilibrate the truncation error, which is induced by M, and the projection error, which is related to the degrees of freedom N_φ and N_ψ, respectively. This implies the choice

$$M \sim \varepsilon^{-\min\{\frac{n_1}{p_1},\frac{n_2}{p_2}\}}, \quad N_\varphi \sim \varepsilon^{-\frac{n_1}{\min\{p_1,r_1\}}}, \quad N_\psi \sim \varepsilon^{-\frac{n_2}{\min\{p_2,r_2\}}}.$$

In particular, the assumption $p_1 > r_1$ or $p_2 > r_2$ implies both, $M \leqslant N_\varphi$ and $M \leqslant N_\psi$. Thus, in view of the cost $M \cdot \max\{N_\varphi, N_\psi\}$, we obtain the following result:

THEOREM 3.4 The number of degrees of freedom, which is needed to approximate a function $f \in H^{p_1,p_2}_{iso}(\Omega_1 \times \Omega_2)$ by the singular value decomposition approach (3.1) to a prescribed accuracy ε, is

$$\mathrm{dof}_{svd}(\varepsilon) \sim \varepsilon^{-\min\{\frac{n_1}{p_1},\frac{n_2}{p_2}\}}\varepsilon^{-\max\{\frac{n_1}{\min\{p_1,r_1\}},\frac{n_2}{\min\{p_2,r_2\}}\}}. \tag{3.18}$$

REMARK 3.2 Note that if $p_1 \leqslant r_1$ and $p_2 \leqslant r_2$, the cost complexity (3.18) also covers the situation when the full tensor product space coincides with Kolmogorov's n-width by means of $M = \min\{N_\varphi, N_\psi\}$. We may hence drop the restriction $p_1 > r_1$ or $p_2 > r_2$ which has been made in the beginning of this subsection. Moreover, the cost complexity (3.18) does not improve if f would possess mixed Sobolev regularity in the sense of $f \in H^{p_1,p_2}_{mix}(\Omega_1 \times \Omega_2)$.

We emphasize that the estimate (3.18) does not include the work to be spent for *computing* the singular values nor the eigenfunctions. Here, a naive approach would result in a cost complexity of order $M \cdot N^2$, where $N := \max\{N_\varphi, N_\psi\}$, the use of fast methods for nonlocal operators would result in an almost linear or even linear complexity per eigenpair. Note that, in any case, at least linear complexity $\mathcal{O}(M \cdot N)$ is required, which is indeed achieved with modern algorithms, see e.g. Dahmen *et al.* (2008), Dai *et al.* (2008), and the references cited therein. Therefore, our analysis is based on the best possible situation for the approximative truncated singular value decomposition and our later comparison will be fair in this respect.

REMARK 3.3 In uncertainty quantification, the truncated singular value decomposition of the random field under consideration is called Karhunen-Loéve approximation. In particular, one has then higher regularity only in the spatial variable, i.e., one considers functions $f \in H^{0,p}_{mix}(\Omega_1 \times \Omega_2)$, where Ω_1 reflects

the sample space and Ω_2 the physical domain. Of course, the truncation estimate (3.13) remains true with $p_1 = 0$ and $p_2 = p$, but an approximation of the eigenfunctions with respect to the sample space is not possible any more. Instead, one prescribes a certain distribution of the random variables which involves a modelling step, see Ghanem & Spanos (1991) or Le Maître & Knio (2010) for details.

4. Sparse grids

Based on the multiscale analyses (2.5) on each individual subdomain, one naturally obtains a second method to approximate functions in tensor product spaces: By choosing complementary spaces

$$W_j^{(i)} = \operatorname{span}\left\{\xi_{j,k}^{(i)} : k \in \nabla_j^{(i)} := \Delta_j^{(i)} \setminus \Delta_{j-1}^{(i)}\right\}, \quad i = 1,2,$$

such that

$$V_j^{(i)} = W_j^{(i)} \oplus V_{j-1}^{(i)}, \quad V_0^{(i)} = W_0^{(i)},$$

we can define the so called *general sparse grid space*, see Bungartz & Griebel (2004) and Griebel & Harbrecht (2013),

$$\widehat{\mathbf{V}}_J^\sigma := \bigoplus_{j_1\sigma + j_2/\sigma \leqslant J} W_{j_1}^{(1)} \otimes W_{j_2}^{(2)} \tag{4.1}$$

where $\sigma > 0$ is a given parameter. Thus, a function $\widehat{f}_J \in \widehat{\mathbf{V}}_J^\sigma$ is represented as

$$\widehat{f}_J(\mathbf{x},\mathbf{y}) = \sum_{j_1\sigma + j_2/\sigma \leqslant J} \sum_{k_1 \in \nabla_{j_1}^{(1)}} \sum_{k_2 \in \nabla_{j_2}^{(2)}} \beta_{(j_1,k_1),(j_2,k_2)} \xi_{j_1,k_1}^{(1)}(\mathbf{x}) \xi_{j_2,k_2}^{(2)}(\mathbf{y}). \tag{4.2}$$

Sparse grids can be constructed via hierarchical bases, interpolets and wavelet-like bases (see e.g. DeVore *et al.* (1998), Griebel & Knapek (2009), Strömberg (1998), and Zenger (1991)) or even directly by finite elements in terms of frames (see e.g. Griebel (1994), Griebel & Oswald (1994), and Harbrecht *et al.* (2008b)). They can also be build from Fourier-, Chebyshev-, Legendre- or similar global polynomial systems, depending on the respective situation. This results in the so-called hyperbolic cross methods. For a survey on sparse grids, we refer the reader to Bungartz & Griebel (2004) and the references therein.

The dimension of the general sparse grid space $\widehat{\mathbf{V}}_J^\sigma$ is essentially equal to the dimension of the finest univariate finite element spaces which enter its construction, i.e., it is essentially equal to the value of $\max\left\{\dim V_{J/\sigma}^{(1)}, \dim V_{J\sigma}^{(2)}\right\}$. Nevertheless, by considering smoothness in terms of mixed Sobolev spaces, its approximation power is essentially the same as in the full tensor product space. Precisely, in accordance with Griebel & Harbrecht (2013), we have the following result:

LEMMA 4.1 The sparse grid space $\widehat{\mathbf{V}}_J^\sigma$ possesses

$$\dim \widehat{\mathbf{V}}_J^\sigma \sim \begin{cases} 2^{J\max\{n_1/\sigma, n_2\sigma\}}, & \text{if } n_1/\sigma \neq n_2\sigma, \\ 2^{Jn_2\sigma} J, & \text{if } n_1/\sigma = n_2\sigma, \end{cases}$$

degrees of freedom. Moreover, for a given function $f \in H_{mix}^{p_1,p_2}(\Omega_1 \times \Omega_2)$ with $0 < p_1 \leqslant r_1$ and $0 < p_2 \leqslant r_2$, there holds the approximation estimate

$$\inf_{\widehat{f}_J \in \widehat{\mathbf{V}}_J^\sigma} \left\| f - \widehat{f}_J \right\|_{L^2(\Omega_1\times\Omega_2)} \lesssim \begin{cases} 2^{-J\min\{p_1/\sigma, p_2\sigma\}} \|f\|_{H_{mix}^{p_1,p_2}(\Omega_1\times\Omega_2)}, & \text{if } p_1/\sigma \neq s_2\sigma, \\ 2^{-Jp_1/\sigma}\sqrt{J}\|f\|_{H_{mix}^{p_1,p_2}(\Omega_1\times\Omega_2)}, & \text{if } p_1/\sigma = p_2\sigma. \end{cases}$$

As shown in Griebel & Harbrecht (2013), the best cost complexity is obtained for the choice $\sigma = \sqrt{n_1/n_2}$ which means that the degrees of freedom in the extremal spaces $V_{J/\sigma}^{(1)}$ and $V_{J\sigma}^{(2)}$ are equilibrated. It especially holds:

THEOREM 4.1 The number of degrees of freedom, which is needed to approximate a function $f \in H_{mix}^{p_1,p_2}(\Omega_1 \times \Omega_2)$ in the sparse grid space $\widehat{\mathbf{V}}_J^\sigma$ with $\sigma = \sqrt{n_1/n_2}$ to a prescribed accuracy ε, is essentially

$$\mathrm{dof}_{sg}^{mix}(\varepsilon) \sim \varepsilon^{-\max\left\{\frac{n_1}{\min\{p_1,r_1\}},\frac{n_2}{\min\{p_2,r_2\}}\right\}} \tag{4.3}$$

In Theorem 4.1, the convergence rate is given for a function $f \in H_{mix}^{p_1,p_2}(\Omega_1 \times \Omega_2)$. Nonetheless, we are also interested in the convergence rate if the smoothness of f is measured in the isotropic Sobolev space $H_{iso}^{p_1,p_2}(\Omega_1 \times \Omega_2)$. Since for all $q_1 = p_1 s$ and $q_2 = p_2(1-s)$ with $s \in [0,1]$ it holds

$$H_{mix}^{q_1,q_2}(\Omega_1 \times \Omega_2) \subset H_{iso}^{p_1,p_2}(\Omega_1 \times \Omega_2),$$

we conclude in view of (4.3) the cost complexity

$$\mathrm{dof}_{sg}^{iso}(\varepsilon) \sim \min_{0<s<1} \varepsilon^{-\max\left\{\frac{n_1}{\min\{sp_1,r_1\}},\frac{n_2}{\min\{(1-s)p_2,r_2\}}\right\}}$$

for functions in $H_{iso}^{p_1,p_2}(\Omega_1 \times \Omega_2)$.

The optimum s is the one that equilibrates the fractions $\frac{n_1}{sp_1}$ and $\frac{n_2}{(1-s)p_2}$, which yields

$$\mathrm{dof}_{sg}^{iso}(\varepsilon) \sim \varepsilon^{-\max\{\frac{n_1}{p_1}+\frac{n_2}{p_2},\frac{n_1}{r_1},\frac{n_2}{r_2}\}}. \tag{4.4}$$

We may hence summarize:

THEOREM 4.2 The number of degrees of freedom, which is needed to approximate a function $f \in H_{iso}^{p_1,p_2}(\Omega_1 \times \Omega_2)$ in the sparse grid space $\widehat{\mathbf{V}}_J^\sigma$ with $\sigma = \sqrt{n_1/n_2}$ to a prescribed accuracy ε, is essentially (4.4).

5. Discussion and concluding remarks

In the present article, we derived the cost complexities of the approximative truncated singular value decomposition and the general sparse grid approach for given, fixed approximation orders r_1 and r_2 and spatial dimensions n_1 and n_2 of the associated finite element methods. We are aware that this setting is different to the one in approximation theory, but it clearly reflects practical needs. For making a decision on which method is preferable, we will compare the two different approximation schemes in case of a bivariate function $f \in H_{iso}^{p_1,p_2}(\Omega_1 \times \Omega_2)$.

According to Theorems 3.4 and 4.2, the cost complexity to ensure a desired approximation error ε is $\mathcal{O}(\varepsilon^{-q})$ with

$$q_{svd} = \min\left\{\frac{n_1}{p_1},\frac{n_2}{p_2}\right\} + \max\left\{\frac{n_1}{\min\{p_1,r_1\}},\frac{n_2}{\min\{p_2,r_2\}}\right\}$$

for the approximation by the singular value decomposition and with

$$q_{sg} = \max\left\{\frac{n_1}{p_1}+\frac{n_2}{p_2},\frac{n_1}{r_1},\frac{n_2}{r_2}\right\}$$

for the sparse grid method.

Straightforward calculation shows that $q_{svd} \geqslant q_{sg}$ for all choices of n_1, n_2, r_1, r_2, p_1, p_2. If $p_1 > r_1$ and $p_2 > r_2$, then it even holds $q_{svd} > q_{sg}$. This means that the approximation by the properly balanced spars grids is superior over the approximation by the singular value decomposition in the function class $H_{iso}^{p_1,p_2}(\Omega_1 \times \Omega_2)$. We emphasize that a change from the isotropic Sobolev space $H_{iso}^{p_1,p_2}(\Omega_1 \times \Omega_2)$ to the anisotropic Sobolev space $H_{mix}^{p_1,p_2}(\Omega_1 \times \Omega_2)$ does only improve the rate q_{sg} in accordance with

$$q_{sg} = \max\left\{\frac{n_1}{\min\{p_1, r_1\}}, \frac{n_2}{\min\{p_2, r_2\}}\right\}$$

while q_{svd} is kept unchanged.

In the situation of low regularity, that is, if $p_1 \leqslant r_1$ and $p_2 \leqslant r_2$, the approximative truncated singular value decomposition coincides with the approximation in the full tensor product space $V_{j_1}^{(1)} \otimes V_{j_2}^{(2)}$, where j_1 and j_2 are related by (2.7). This full tensor product space is known to realize Kolmogorov's n-width $\varepsilon^{-\frac{n_1}{p_1}-\frac{n_2}{p_2}}$ for Sobolev balls in the space $H_{iso}^{p_1,p_2}(\Omega_1 \times \Omega_2)$. The sparse grid approach would essentially also give Kolmogorov's n-width, but has in general larger constants.

In case of the sparse grid approach, we envision further improvements by the use of local adaptivity, which would further increase its performance.

In case of the singular value decomposition, the truncation length is determined by the smoothness of the function under consideration and is thus fixed. Therefore, improvements for the truncated singular value decomposition can only be achieved by a more efficient representation of the eigenfunctions.

Future work is needed to study the cost complexity in the higher dimensional setting. Especially, the comparison of the recursive application of the singular value decomposition in a hierarchical tensor format and the sparse grid approach would be relevant for practical application.

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